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Classical continuous orthogonal polynomials

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Definitions

Activity 1

Let n be a non-negative integer, $x \in (-1, 1)$.We consider

$$T_n(x) = \cos(nArccos(x)).$$

$$T_0(x) = 1, \quad T_1(x) = x. \tag{1}$$

$$T_{n+1}(x) + T_{n-1}(x) = 2xT_n(x). \tag{2}$$

Using (2), T_n is a polynomial of degree exactly n .

Definitions

Activity 1

1. Evaluate in function of the integers n and m the quantity:

$$I_{n,m} = \int_{-1}^{1} T_n(x)T_m(x)\rho(x)dx, \;\; \texttt{with} \;\; \rho(x) = \frac{1}{\sqrt{1-x^2}} \tag{3}$$

2. Evaluate the quantity:

$$A := ((1-x^2)\rho(x))' + x\rho(x) \tag{4}$$

Maple computation

Definitions

1. For any non-negative n, T_n is a polynomial of degree exactly n.
2. $$I_{n,m} = \int_{-1}^{1} T_n(x)T_m(x)\rho(x)dx = 0, \ \texttt{for} \ \ n \neq m. \tag{5}$$
3. $$I_{n,n} = \int_{-1}^{1} T_n(x)T_n(x)\rho(x)dx \neq 0. \tag{6}$$

$(T_n)_{n\in\mathbb{N}}$ is said to be an orthogonal polynomials sequence.

Definitions

Definition 1

Let ρ be a positive continuous function defined on the interval (a, b); $a < b$. ρ is called weight function if there exists (μ_n) a sequence of complex numbers such that

$$\mu_n = \int_a^b x^n \rho(x) dx; \quad \forall n \in \mathbb{N}.$$

Definitions

Definition 2

Let ρ be a weight function defined on (a, b) . A set of polynomials (P_n) is said to be an orthogonal polynomials sequence (OPS) with respect to ρ if, for each non-negative integer n, $degree(P_n) = n$ and

$$\int_a^b P_n(x)P_m(x)\rho(x)dx = k_n\delta_{nm}, \quad (k_n \neq 0), \quad \forall n, m = 0, 1, \cdots .$$

Definitions

Definition 3

The classical orthogonal polynomials sequence are the only OPS such that the corresponding weight function ρ satisfies a differential equation

$$(\phi(x)\rho(x))' = \psi(x)\rho(x), \tag{7}$$

with

$$degree(\phi) \leq 2 \text{ and } degree(\psi) = 1$$

and the bounded condition

$$\lim_{x \longrightarrow a} x^m \phi(x)\rho(x) = 0, \quad \lim_{x \longrightarrow b} x^m \phi(x)\rho(x) = 0. \tag{8}$$

orthogonality of the derivatives

Activity 2

1. Evaluate in function of the integers n and m the quantity:

$$J_{n,m} = \int_{-1}^{1} T'_{n+1}(x) T'_{m+1}(x) \phi(x) \rho(x) dx, \quad \texttt{with} \quad \phi(x) = 1 - x^2. \tag{9}$$

2. conclude that (T'_{n+1}) is an orthogonal polynomials sequence.

Maple computation

orthogonality of the derivatives

Activity 2

1. T'_{n+1} is a polynomials of degree exactly n.
2. $J_{n,m} = 0, \quad m \neq n.$
3. $J_{n,n} \neq 0$

The sequence of the derivatives (T'_{n+1}) is an orthogonal polynomials sequence with respect to the weight function $\rho_1(x) = \phi(x)\rho(x)$.

orthogonality of the derivatives

Theorem 1

If (P_n) is a classical orthogonal polynomials sequence with respect to the weight function ρ defined on (a, b), then, (P'_{n+1}) is an orthogonal polynomials sequence with respect to the weight function

$$\rho_1(x) = \phi(x)\rho(x).$$

orthogonality of the derivatives

Proof of Theorem 1

Let (P_n) be a classical orthogonal polynomials sequence with respect to the weight function ρ defined on (a, b).
Let $m, n \in \mathbb{N}$ such that $m < n$. We have

$$
\begin{aligned}
0 &= \int_a^b P_n(x)x^{m-1}\psi(x)\rho(x)dx = \int_a^b P_n(x)x^{m-1}(\phi(x)\rho(x))'dx \\
&= P_n(x)x^{m-1}\phi(x)\rho(x)\Big|_a^b - \int_a^b (P_n(x)x^{m-1})'\phi(x)\rho(x)dx \\
&= -\int_a^b P_n'(x)x^{m-1}\phi(x)\rho(x)dx - \int_a^b P_n(x)x^{m-2}\phi(x)\rho(x)dx
\end{aligned}
$$

orthogonality of the derivatives

Proof of Theorem 1

$$\int_a^b P_n'(x)x^{m-1}\phi(x)\rho(x)dx = 0, \quad m < n$$

$$\int_a^b P_n'(x)x^{n-1}\phi(x)\rho(x)dx \neq 0.$$

(P_{n+1}') is an orthogonal polynomials sequence with respect to the weight function $\rho_1(x) = \phi(x)\rho(x)$.

In addition

$$(\phi(x)\rho_1(x))' = (\phi'(x) + \psi(x))\rho_1(x).$$

(P_{n+1}') is then also classical.

orthogonality of the derivatives

Theorem 2

Let (P_n) be a classical orthogonal polynomials sequence with respect to the weight function ρ defined on (a, b) and m a fixed non-negative integer.
$(\frac{d^m P_{n+m}}{dx^m})$ is a classical orthogonal polynomials sequence with respect to the weight function $\rho_m(x) = (\phi(x))^m \rho(x)$.
We have also

$$(\phi(x)\rho_m(x))' = (m\phi'(x) + \psi(x))\rho_m(x) = \psi_m(x)\rho_m(x).$$

Second order differential equation

Activity 3

We consider

$$T_n(x) = \cos(nArccos(x)).$$

1. Evaluate

$$T_n'(x),\ T_n''(x).$$

2. Verify that

$$(1 - x^2)T_n''(x) - xT_n'(x) + n^2 T_n(x) = 0.$$

Maple computation

Second order differential equation

Theorem 3

Let (P_n) be a classical orthogonal polynomials sequence with respect to the weight function ρ defined on (a, b).
For all non-negative integer n, we have

$$\phi(x)P_n''(x) + \psi(x)P_n'(x) + \lambda_n P_n(x) = 0 \tag{10}$$

where $\lambda_n = -n\psi' - \frac{n(n-1)}{2}\phi''$.

Second order differential equation

Proof of Theorem 3

Let (P_n) be a classical orthogonal polynomials sequence with respect to the weight function ρ defined on (a, b).
Let $m, n \in \mathbb{N}$ such that $m < n$. We have

$$\begin{aligned} 0 &= \int_a^b P_n'(x)(x^m)'\phi(x)\rho(x)dx \\ &= P_n'(x)x^m\phi(x)\rho(x)\Big|_a^b - \int_a^b (\phi(x)\rho(x)P_n'(x))'\, x^m dx \\ &= -\int_a^b (\phi(x)P_n''(x) + \psi(x)P_n'(x))\, x^m \rho(x)dx \end{aligned}$$

Second order differential equation

Proof of Theorem 3

$$\int_a^b \left(\phi(x)P_n''(x) + \psi(x)P_n'(x)\right) x^m \rho(x)dx = 0, \quad m < n$$

$$\int_a^b \left(\phi(x)P_n''(x) + \psi(x)P_n'(x)\right) x^n \rho(x)dx \neq 0.$$

$$\phi P_n'' + \psi P_n'$$

is a polynomials of degree n.

Second order differential equation

Proof of Theorem 3

There exists a constant λ_n such that

$$\phi(x)P_n''(x) + \psi(x)P_n'(x) + \lambda_n P_n(x) = 0. \quad (11)$$

Comparing the coefficients of x^n in (11), we have

$$\lambda_n = -n\psi' - \frac{n(n-1)}{2}\phi''.$$

Rodrigues Formula

Let (P_n)be a classical orthogonal polynomials sequence with respect to the weight function ρ . We have

$$\begin{aligned} -\lambda_n \rho(x) P_n(x) &= (\phi(x)\rho(x)P_n'(x))' \\ -\mu_{n,1}\rho_1(x)P_n'(x) &= ((\phi(x))^2\rho(x)P_n''(x))'. \end{aligned}$$

Then

$$\rho(x)P_n(x) = \frac{1}{\lambda_n \mu_{n,1}} \left(\phi(x))^2 \rho(x) P_n''(x)\right)''$$

Continuing the process, we obtain

$$P_n(x) = \frac{A_n}{\rho(x)} \frac{d^n}{dx^n} \left((\phi(x))^n \rho(x)\right). \tag{12}$$

(12) is called the Rodrigues Formula for the family (P_n).

Classification of classical orthogonal polynomials

Classical orthogonal polynomials sequence is the one that the corresponding weight function ρ satisfies the Pearson type equation

$$(\rho(x)\phi(x))' = \psi(x)\rho(x) \tag{13}$$

where ϕ is a polynomial of degree at most two and ψ is the first degree polynomial. (13) is equivalent to

$$\frac{\rho'(x)}{\rho(x)} = \frac{\psi(x)+\phi'(x)}{\phi(x)}.$$

By searching the solution of this differential equation, we obtain

$$\rho(x) = \exp\left(\int \frac{\psi(x)+\phi'(x)}{\phi(x)}dx\right).$$

We will give the classification in term of the degree of the polynomial ϕ.

Jacobi

If $deg(\phi) = 2$ and ϕ has two distinct zeros, then there exist a, b, α and β such that

$$\frac{\psi(x) + \phi'(x)}{\phi(x)} = \frac{\alpha}{b - x} + \frac{\beta}{x - a}.$$

If we consider $t = \frac{2x-a-b}{b-a}$, then the interval of orthogonality (a, b) becomes $(-1, 1)$, $\rho(t) = (1 - t)^{\alpha}(1 + t)^{\beta}$, $\phi(t) = (1 - t^2)$ and we obtain $\psi(t) = -(\alpha + \beta + 2)t + \beta - \alpha$, with $\alpha, \beta > -1$.

Jacobi

We obtain the Jacobi polynomials denoted $(P_n^{\alpha,\beta})$.
$\rho(x) = (1-x)^{\alpha}(1+x)^{\beta},\ \alpha > -1,\ \beta > -1.$ $\phi(x) = 1 - x^2$,
$\psi(x) = \beta - \alpha - (\alpha + \beta + 2)x$ and $\lambda_n = n(n + \alpha + \beta + 1)$.
The second order differential equation satisfies by the Jacobi polynomial is

$$(1-x^2)y''(x) + (\beta - \alpha - (\alpha+\beta+2)x)\, y'(x) + n(n+\alpha+\beta+1)y(x) = 0,$$
$$y(x) = P_n^{\alpha,\beta}(x).$$

Rodrigues formula

$$P_n^{\alpha,\beta}(x) = \frac{(-1)^n}{2^n n!(1-x)^{\alpha}(1+x)^{\beta}} \left((1-x)^{\alpha+n}(1+x)^{\beta+n}\right)^{(n)}.$$

Laguerre

If $deg(\phi) = 1$, the there exist a, b and α such that

$$\frac{\psi(x) + \phi'(x)}{\phi(x)} = \frac{\alpha}{x - a} + b.$$

If we use the change of variable $t = \begin{cases} -x + a \\ x - a \end{cases}$, then we take the interval of orthogonality $(0, +\infty)$, $\rho(t) = t^{\alpha} e^{-t}$, $\phi(t) = t$ and we obtain $\psi(t) = -t + \alpha + 1$, with $\alpha > -1$.

Laguerre

We obtain the Laguerre polynomials denoted (L_n^{α}).
We consider $\phi(x) = x,\ x \in]0, +\infty[\ \rho(x) = x^{\alpha} e^{-x},\ \alpha > -1,$
$\psi(x) = \alpha + 1 - x$ and $\lambda_n = n$.
The second order differential equation satisfies by the Laguerre polynomial is

$$xy''(x) + (\alpha + 1 - x)\, y'(x) + ny(x) = 0,\ \ y(x) = L_n^{\alpha}(x).$$

Rodrigues formula

$$L_n^{\alpha}(x) = \frac{1}{n! x^{\alpha} e^{-x}} \left(x^{\alpha+n} e^{-x} \right)^{(n)}.$$

Hermite

If $deg(\phi) = 0$, the there exist a and b such that

$$\frac{\psi(x) + \phi'(x)}{\phi(x)} = ax + b.$$

If we use the change of variable $t = -\psi(x)$, then we take the interval of orthogonality $(-\infty, +\infty)$, $\rho(t) = e^{-t^2}$, $\phi(t) = 1$ and we obtain $\psi(t) = -2t$.

Hermite

We obtain the Hermite polynomials denoted (H_n).
$(a, b) = \mathbb{R}$, $\rho(x) = e^{-x^2}$, $\phi(x) = 1$, $\psi(x) = -2x$ and $\lambda_n = 2n$.
The second order differential equation satisfies by the Hermite polynomial is

$$H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0.$$

Rodrigues formula

$$H_n(x) = \frac{(-1)^n}{e^{-x^2}} \left(e^{-x^2}\right)^{(n)}.$$

The following theorem gives some characterization properties of classical continuous orthogonal polynomials.

Characterization theorem of classical orthogonal polynomials

Let (P_n) be the monic orthogonal polynomials sequence with respect to the weight function ρ on the interval (a, b) and $Q_{n,m}$ the monic polynomial of degree n defined by

$$Q_{n,m}(x) = \frac{n!}{(n+m)!}\frac{d^m}{dx^m}P_{n+m}(x), \quad Q_{n,0}(x) = P_n(x).$$

The following properties are equivalent:

a) There exist two polynomials, ϕ of degree at most two and ψ of degree one, such that

$$(\phi(x)\rho(x))' = \psi(x)\rho(x).$$

Characterization theorem of classical orthogonal

b) There exist two polynomials, ϕ of degree at most two and ψ of degree one, such that for any integer m,

$$\left[(\phi(x))^{m+1}\rho(x)\right]' = (\psi(x) + m\phi'(x))(\phi(x))^m\rho(x)$$

$$\int_a^b Q_{j,m}(x)Q_{n,m}(x)(\phi(x))^m\rho(x)dx = k_n\delta_{j,n}, \quad k_n \neq 0; \quad \forall j, n \in \mathbb{N}.$$

c) There exist two polynomials, ϕ of degree at most two and ψ of degree one, such that for any integer m, the following second-order differential equation holds

$$\phi(x)Q''_{n,m}(x) + (\psi(x) + m\phi'(x))Q'_{n,m}(x) + \mu_{n,m}Q_{n,m}(x) = 0$$

Characterization theorem of classical orthogonal

with the constant $\mu_{n,m}$ given by

$$\mu_{n,m} = -n\left[\psi' + (2m+n-1)\frac{\phi''}{2}\right]$$

d) There exist two polynomials, ϕ of degree at most two and ψ of degree one, such that for any integer m, the following relation holds

$$Q_{n,m}(x) = \frac{A_{n,m}}{(\phi(x))^m \rho(x)} \frac{d^n}{dx^n}\left((\phi(x))^{m+n}\rho(x)\right)$$

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Thank you for your attention

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