Classical orthogonal polynomials of a discrete and a q-discrete variable

P. Njionou Sadjang*
pnjionou@easy-maths.org

AIMS-VOLKSWAGEN STIFTUNG WORKSHOP
on
Introduction to orthogonal polynomials and applications

* Faculty of Industrial Engineering, University of Douala

Douala, October 2018

Table of Contents

Table of Contents

Table of Contents

Table of Contents

Introduction

Classical orthogonal polynomials can be defined by giving

1. the weight function
2. the hypergeometric (or q-hypergeometric representation)
3. the second order differential (difference, q-difference) equation
4. the three-term recurrence relation (Farvard theorem)
5. the Rodrigues representation
6. a generating function

Introduction

Classical orthogonal polynomials can be defined by giving

1. the weight function
2. the hypergeometric (or q-hypergeometric representation)
3. the second order differential (difference, q-difference) equation
4. the three-term recurrence relation (Farvard theorem)
5. the Rodrigues representation
6. a generating function

Introduction

Classical orthogonal polynomials can be defined by giving

1. the weight function
2. the hypergeometric (or q-hypergeometric representation)
3. the second order differential (difference, q-difference) equation
4. the three-term recurrence relation (Farvard theorem)
5. the Rodrigues representation
6. a generating function

Introduction

Classical orthogonal polynomials can be defined by giving

1. the weight function
2. the hypergeometric (or q-hypergeometric representation)
3. the second order differential (difference, q-difference) equation
4. the three-term recurrence relation (Farvard theorem)
5. the Rodrigues representation
6. a generating function

Introduction

Classical orthogonal polynomials can be defined by giving

1. the weight function
2. the hypergeometric (or q-hypergeometric representation)
3. the second order differential (difference, q-difference) equation
4. the three-term recurrence relation (Farvard theorem)
5. the Rodrigues representation
6. a generating function

Introduction

Classical orthogonal polynomials can be defined by giving

1. the weight function
2. the hypergeometric (or q-hypergeometric representation)
3. the second order differential (difference, q-difference) equation
4. the three-term recurrence relation (Farvard theorem)
5. the Rodrigues representation
6. a generating function

Introduction

We present

- Orthogonal polynomials of discrete variable.

$$\frac{d}{dx} \longrightarrow \Delta \quad \text{or} \quad \nabla$$

and

$$\int \longrightarrow \sum$$

also

$$x^n \longrightarrow \text{ Pochhammer } (x)_n \text{ or the falling factorial } (x)_{[n]}$$

Introduction

We present

- Orthogonal polynomials of discrete variable.

$$\frac{d}{dx} \longrightarrow \Delta \quad \text{or} \quad \nabla$$

and

$$\int \longrightarrow \sum$$

also

$$x^n \longrightarrow \text{ Pochhammer } (x)_n \text{ or the falling factorial } (x)_{[n]}$$

Introduction

We present

- Orthogonal polynomials of discrete variable.

$$\frac{d}{dx} \longrightarrow \Delta \quad \text{or} \quad \nabla$$

and

$$\int \longrightarrow \sum$$

also

$$x^n \longrightarrow \text{ Pochhammer } (x)_n \text{ or the falling factorial } (x)_{[n]}$$

Introduction

- Orthogonal polynomials of the q-discrete variable.

$$\frac{d}{dx} \longrightarrow D_q$$

the integral is in most of the cases replaced by the q-integral and the power x^n is somtimes replaced by the q-Pochhammer or the q-power.

discrete case, basic definition

Discrete orthogonal polynomials

Forward difference

Let f be a function of the variable x. The forward and the backward operators Δ and ∇ are, respectively, defined by:

$$\Delta f(x) = f(x+1) - f(x), \qquad \nabla f(x) = f(x) - f(x-1).$$

For $m \in \mathbb{N} = \{1, 2, 3, \ldots\}$, one sets

$$\Delta^{m+1} f(x) = \Delta(\Delta^m f(x)).$$

Pochhammer symbol

The Pochhammer symbol or shifted factorial is defined by

$$\begin{cases} (a)_0 := 1 \\ (a)_n = a(a+1)(a+2)\cdots(a+n-1), \quad a \neq 0 \quad n = 1, 2, 3, \ldots. \end{cases}$$

Discrete orthogonal polynomials

Forward difference

Let f be a function of the variable x. The forward and the backward operators Δ and ∇ are, respectively, defined by:

$$\Delta f(x) = f(x+1) - f(x), \qquad \nabla f(x) = f(x) - f(x-1).$$

For $m \in \mathbb{N} = \{1, 2, 3, \dots\}$, one sets

$$\Delta^{m+1} f(x) = \Delta(\Delta^m f(x)).$$

Pochhammer symbol

The Pochhammer symbol or shifted factorial is defined by

$$\begin{cases} (a)_0 := 1 \\ (a)_n = a(a+1)(a+2)\cdots(a+n-1), \quad a \neq 0 \quad n = 1, 2, 3, \dots. \end{cases}$$

Discrete orthogonal polynomials

The falling factorial

The following notation (falling factorial) will also be used:

$$a_{[0]} := 1 \quad \text{and} \quad a_{[n]} = a(a-1)(a-2)\cdots(a-n+1), \quad n = 1, 2, 3, \ldots .$$

It should be noted that the Pochhammer symbol and the falling factorial are linked as follows:

$$(-a)_n = (-1)^n a_{[n]}.$$

In fact,

$$\begin{aligned}(-a)_n &= (-a)(-a+1)(-a+2)\ldots(-a+n-1)\\ &= (-1)^n a(a-1)(a-2)\ldots(a-n+1) = (-1)^n (a)_{[n]}.\end{aligned}$$

Discrete orthogonal polynomials

The hypergeometric series

The hypergeometric series ${}_rF_s$ is defined by

$${}_rF_s\left(\begin{array}{c} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{array}\middle| z\right) := \sum_{n=0}^{\infty} \frac{(a_1, \cdots, a_r)_n}{(b_1, \cdots, b_s)_n} \frac{z^n}{n!},$$

where

$$(a_1, \ldots, a_r)_n = (a_1)_n \cdots (a_r)_n.$$

Discrete orthogonal polynomials

An example of a summation formula for the hypergeometric series is given by the binomial theorem

$$ {}_1F_0\left(\begin{array}{c} a \\ - \end{array} \middle| -z \right) = \sum_{n=0}^{\infty} \binom{a}{n} z^n = (1+z)^a, \quad |z| < 1, $$

where

$$ \binom{a}{n} = \frac{(-1)^n}{n!}(-a)_n = \frac{(a)_{[n]}}{n!}. $$

Discrete orthogonal polynomials

Discrete orthogonal polynomials

A polynomial set

$$y(x) = p_n(x) = k_n x^n + \dots \quad (n \in \mathbb{N}_0 = \{0, 1, 2, \dots\}, \quad k_n \neq 0) \qquad (1)$$

is a family of discrete classical orthogonal polynomials (also known as the Hahn class) if it is the solution of a difference equation of the type

$$\sigma(x)\Delta\nabla y(x) + \tau(x)\Delta y(x) + \lambda_n y(x) = 0. \qquad (2)$$

They are known to satisfy the Pearson-type equation

$$\Delta[\sigma(x)\rho(x)] = \tau(x)\rho(x) \qquad (3)$$

where $\rho(x)$ is the discrete weight function for which the p_n's are orthogonal.

Discrete orthogonal polynomials

Orthogonality of the difference

Let $(p_n(x))_n$ be a family of classical discrete orthogonal polynomials satisfying the Pearson-type difference equation (3), then the family $(\Delta p_{n+1}(x))_n$ is orthogonal with respect to the weight function ρ_1 defined by $\rho_1(x) = \sigma(x+1)\rho(x+1)$.

Orthogonality of the k-th difference

Let $(p_n(x))_n$ be a family of classical discrete orthogonal polynomials satisfying the Pearson-type difference equation (3), then the family $(\Delta^k p_{n+k}(x))_n$ is orthogonal with respect to the weight function ρ_k defined by $\rho_k(x) = \rho(x+k)\prod_{j=1}^{k}\sigma(x+j)$.

Rodrigues formula

Discrete orthogonal polynomials

Rodrigues formula

Let $(P_n(x))_n$ be a family of classical discrete orthogonal polynomials. If we set $P_{m,n}(x) = \Delta^m P_n(x)$ $(m \leq n)$, then

$$P_{m,n}(x) = \frac{A_{mn}B_n}{\rho_m(x)} \nabla^{n-m}[\rho_n(x)] \tag{4}$$

with

$$\begin{aligned} A_{mn} &= \frac{n!}{(n-m)!} \prod_{k=0}^{m-1} \left(\tau' + \frac{n+k-1}{2}\sigma"\right); \\ A_{0n} &= 1; \\ B_n &= \frac{\Delta^n P_n(x)}{A_{nn}} = \frac{1}{A_{nn}} P_n^{(n)}(x). \end{aligned}$$

Discrete orthogonal polynomials

Rodrigues formula

For $m = 0$ in (4) it follows that,

$$P_n(x) = P_{0n}(x) = y_n(x) = \frac{B_n}{\rho(x)} \nabla^n[\rho_n(x)]. \tag{5}$$

Discrete orthogonal polynomials

Now, we look for the solutions of the difference equation

$$\sigma(x)\Delta\nabla y(x) + \tau(x)\Delta y(x) + \lambda y(x) = 0. \tag{6}$$

where σ, τ et λ have the for

- $\sigma(x) = \sigma_2 x^2 + \sigma_1 x + \sigma_0 \quad \sigma_0, \sigma_1, \sigma_2 \in \mathbb{R}$;
- $\tau(x) = \tau_1 x + \tau_0 \quad \tau_1 \in \mathbb{R}^*,\ \tau_0 \in \mathbb{R}$.
- λ is a constant.

Since the suitable polynomial basis for the forward (or the backward) difference is the falling factorial, we write the solution has

$$y(x) = \sum_{k=0}^{\infty} \alpha_k (x)_{[k]}. \tag{7}$$

Discrete orthogonal polynomials

Result 1

The infinite series $y(x)$ defined by (7) is solution of (6) if and only if the α_i's are solutions of the recurrence equation

$$R\alpha_{i+2} + S\alpha_{i+1} + T\alpha_i = 0 \tag{8}$$

where

$$\begin{aligned} R &= \sigma_2 i^4 + (\tau_1 + \sigma_1 + 5\sigma_2)i^3 + (4\tau_1 + 9\sigma_2 + 4\sigma_1 + \sigma_0 + \tau_0)i^2 \\ &\quad + (3\sigma_0 + 7\sigma_2 + 3\tau_0 + 5\tau_1 + 5\sigma_1)i + 2\tau_1 + 2\tau_0 + 2\sigma_1 + 2 \\ &\quad \sigma_0 + 2\sigma_2; \\ S &= 2\sigma_2 i^3 + (3\sigma_2 + \sigma_1 + 2\tau_1)i^2 + (\sigma_2 + \sigma_1 + 3\tau_1 + \tau_0 + \lambda)i + \\ &\quad \tau_1 + \tau_0 + \lambda; \\ R &= \sigma_2 i^2 + (-\sigma_2 + \tau_1)i + \lambda. \end{aligned}$$

Discrete orthogonal polynomials

In the previous statement, the polynomial σ is of the form $\sigma(x) = \sigma_2 x^2 + \sigma_1 x + \sigma_0$ where σ_0 is any real number. For the classical orthogonal polynomials, σ is of the form $\sigma(x) = \sigma_2 x^2 + \sigma_1 x$. The previous Theorem reduces to

Result 2

If $\sigma(x) = \sigma_2 x^2 + \sigma_1 x$, the serie y defined by (7) is solution of (6) if and only if the α_i's verify the recurrence relation

$$(i+1)[i(\sigma_2 i + \sigma_1) + (i\tau_1 + \tau_0)]\alpha_{i+1} - [\sigma_2 i(i-1) + \tau_1 i + \lambda]\alpha_i = 0. \quad (9)$$

Next, we obtain the hypergeometric representations of the four classical discrete orthogonal polynomials.

Discrete orthogonal polynomials

- *The Charlier polynomials*

  The difference equation is

$$x\Delta\nabla y(x) + (c-x)\Delta y(x) + \lambda y(x) = 0.$$

Then,

$$\sigma_2 = 0, \quad \sigma_1 = 1, \quad \tau_1 = -1, \quad \text{et} \quad \tau_0 = c.$$

The recurrence relation for the α_i's becomes

$$\alpha_{i+1} = \frac{1}{c}\frac{i-n}{i+1}\alpha_i.$$

Taking the normalization $\alpha_0 = 1$, we get:

$$\alpha_i = \left(\frac{1}{c}\right)^i \frac{(-n)_i}{i!}.$$

Finally we obtain the hypergeometric representation

$$C_n(x,c) = \sum_{k=0}^{n} \left(\frac{1}{c}\right)^k \frac{(-n)_k}{k!}(x)_{[k]} = {}_2F_0\left(\begin{array}{c} -n,-x \\ - \end{array} \middle| -\frac{1}{c}\right).$$

Discrete orthogonal polynomials

- *The Charlier polynomials*
  The difference equation is

$$x\Delta\nabla y(x) + (c - x)\Delta y(x) + \lambda y(x) = 0.$$

Then,

$$\sigma_2 = 0, \quad \sigma_1 = 1, \quad \tau_1 = -1, \quad \text{et} \quad \tau_0 = c.$$

The recurrence relation for the α_i's becomes

$$\alpha_{i+1} = \frac{1}{c}\frac{i-n}{i+1}\alpha_i.$$

Taking the normalization $\alpha_0 = 1$, we get:

$$\alpha_i = \left(\frac{1}{c}\right)^i \frac{(-n)_i}{i!}.$$

Finally we obtain the hypergeometric representation

$$C_n(x,c) = \sum_{k=0}^{n} \left(\frac{1}{c}\right)^k \frac{(-n)_k}{k!}(x)_{[k]} = {}_2F_0\left(\begin{matrix} -n, -x \\ - \end{matrix} \middle| -\frac{1}{c}\right).$$

Discrete orthogonal polynomials

- *The Charlier polynomials*
  The difference equation is

$$x\Delta\nabla y(x) + (c-x)\Delta y(x) + \lambda y(x) = 0.$$

Then,

$$\sigma_2 = 0, \quad \sigma_1 = 1, \quad \tau_1 = -1, \quad \text{et} \quad \tau_0 = c.$$

The recurrence relation for the α_i's becomes

$$\alpha_{i+1} = \frac{1}{c}\frac{i-n}{i+1}\alpha_i.$$

Taking the normalization $\alpha_0 = 1$, we get:

$$\alpha_i = \left(\frac{1}{c}\right)^i \frac{(-n)_i}{i!}.$$

Finally we obtain the hypergeometric representation

$$C_n(x,c) = \sum_{k=0}^{n} \left(\frac{1}{c}\right)^k \frac{(-n)_k}{k!}(x)_{[k]} = {}_2F_0\left(\begin{matrix} -n, -x \\ - \end{matrix} \middle| -\frac{1}{c}\right).$$

Discrete orthogonal polynomials

- *The Charlier polynomials*
  The difference equation is

$$x\Delta\nabla y(x) + (c-x)\Delta y(x) + \lambda y(x) = 0.$$

Then,

$$\sigma_2 = 0, \quad \sigma_1 = 1, \quad \tau_1 = -1, \quad \text{et} \quad \tau_0 = c.$$

The recurrence relation for the α_i's becomes

$$\alpha_{i+1} = \frac{1}{c}\frac{i-n}{i+1}\alpha_i.$$

Taking the normalization $\alpha_0 = 1$, we get:

$$\alpha_i = \left(\frac{1}{c}\right)^i \frac{(-n)_i}{i!}.$$

Finally we obtain the hypergeometric representation

$$C_n(x,c) = \sum_{k=0}^{n} \left(\frac{1}{c}\right)^k \frac{(-n)_k}{k!}(x)_{[k]} = {}_2F_0\left(\begin{matrix} -n, -x \\ - \end{matrix} \middle| -\frac{1}{c}\right).$$

Discrete orthogonal polynomials

- *The Charlier polynomials*
  The difference equation is

$$x\Delta\nabla y(x) + (c-x)\Delta y(x) + \lambda y(x) = 0.$$

Then,

$$\sigma_2 = 0, \quad \sigma_1 = 1, \quad \tau_1 = -1, \quad \text{et} \quad \tau_0 = c.$$

The recurrence relation for the α_i's becomes

$$\alpha_{i+1} = \frac{1}{c}\frac{i-n}{i+1}\alpha_i.$$

Taking the normalization $\alpha_0 = 1$, we get:

$$\alpha_i = \left(\frac{1}{c}\right)^i \frac{(-n)_i}{i!}.$$

Finally we obtain the hypergeometric representation

$$C_n(x,c) \quad = \quad \sum_{k=0}^{n}\left(\frac{1}{c}\right)^k \frac{(-n)_k}{k!}(x)_{[k]} = {}_2F_0\left(\begin{array}{c} -n, -x \\ - \end{array} \middle| -\frac{1}{c}\right).$$

Discrete orthogonal polynomials

- *Meixner polynomials*

  The difference equation is

$$x\Delta\nabla y(x) + ((\mu - 1)x + \gamma\mu)\Delta y(x) + \lambda y(x) = 0.$$

Then,

$$\sigma_2 = 0, \quad \sigma_1 = 1, \quad \tau_1 = \mu - 1 \quad \text{et} \quad \tau_0 = \gamma\mu.$$

The recurrence relation for the α_i's becomes

$$\alpha_{i+1} = \frac{(n-i)(\mu-1)}{(i+1)\mu(i+\gamma)}\alpha_i = \left(1 - \frac{1}{\mu}\right)\frac{(n-i)}{(i+1)}\frac{1}{i+\gamma}\alpha_i.$$

Finally we obtain the hypergeometric representation

$$M_n(x,\gamma,\mu) \quad = \quad {}_2F_1\left(\begin{array}{c} -n, -x \\ \gamma \end{array} \middle| 1 - \frac{1}{\mu}\right).$$

Discrete orthogonal polynomials

- *Meixner polynomials*
  The difference equation is

$$x\Delta\nabla y(x) + ((\mu - 1)x + \gamma\mu)\Delta y(x) + \lambda y(x) = 0.$$

Then,

$$\sigma_2 = 0, \quad \sigma_1 = 1, \quad \tau_1 = \mu - 1 \quad \text{et} \quad \tau_0 = \gamma\mu.$$

The recurrence relation for the α_i's becomes

$$\alpha_{i+1} = \frac{(n-i)(\mu-1)}{(i+1)\mu(i+\gamma)}\alpha_i = \left(1 - \frac{1}{\mu}\right)\frac{(n-i)}{(i+1)}\frac{1}{i+\gamma}\alpha_i.$$

Finally we obtain the hypergeometric representation

$$M_n(x,\gamma,\mu) = {}_2F_1\left(\begin{matrix} -n, -x \\ \gamma \end{matrix} \,\middle|\, 1 - \frac{1}{\mu}\right).$$

Discrete orthogonal polynomials

- *Meixner polynomials*
  The difference equation is

$$x\Delta\nabla y(x) + ((\mu - 1)x + \gamma\mu)\Delta y(x) + \lambda y(x) = 0.$$

Then,

$$\sigma_2 = 0, \quad \sigma_1 = 1, \quad \tau_1 = \mu - 1 \quad \text{et} \quad \tau_0 = \gamma\mu.$$

The recurrence relation for the α_i's becomes

$$\alpha_{i+1} = \frac{(n-i)(\mu-1)}{(i+1)\mu(i+\gamma)}\alpha_i = \left(1 - \frac{1}{\mu}\right)\frac{(n-i)}{(i+1)}\frac{1}{i+\gamma}\alpha_i.$$

Finally we obtain the hypergeometric representation

$$M_n(x,\gamma,\mu) = {}_2F_1\left(\begin{array}{c} -n, -x \\ \gamma \end{array} \middle| 1 - \frac{1}{\mu}\right).$$

Discrete orthogonal polynomials

- *Meixner polynomials*
  The difference equation is

$$x\Delta\nabla y(x) + ((\mu - 1)x + \gamma\mu)\Delta y(x) + \lambda y(x) = 0.$$

Then,

$$\sigma_2 = 0, \quad \sigma_1 = 1, \quad \tau_1 = \mu - 1 \quad \text{et} \quad \tau_0 = \gamma\mu.$$

The recurrence relation for the α_i's becomes

$$\alpha_{i+1} = \frac{(n-i)(\mu-1)}{(i+1)\mu(i+\gamma)}\alpha_i = \left(1 - \frac{1}{\mu}\right)\frac{(n-i)}{(i+1)}\frac{1}{i+\gamma}\alpha_i.$$

Finally we obtain the hypergeometric representation

$$M_n(x,\gamma,\mu) \quad = \quad {}_2F_1\left(\begin{array}{c} -n, -x \\ \gamma \end{array} \middle| \, 1 - \frac{1}{\mu}\right).$$

Discrete orthogonal polynomials

- *Kravchuk polynomials*
  The difference equation is

$$x\Delta\nabla y(x) + \Big(-\frac{1}{q}x + \frac{Np}{q}\Big)\Delta y(x) + \lambda y(x) = 0.$$

Then,

$$\sigma_2 = 0, \quad \sigma_1 = 1, \quad \tau_1 = -\frac{1}{q} \quad \text{et} \quad \tau_0 = \frac{Np}{q}.$$

The recurrence relation for the α_i's becomes

$$\alpha_{i+1} = -\frac{(n-i)(\mu-1)}{p(i+1)(i+N)}\alpha_i = -\frac{(n-i)}{(i+1)}\frac{1}{p}\frac{1}{i+N}\alpha_i.$$

Finally we obtain the hypergeometric representation

$$\begin{aligned} K_n(x,p,N) &= \sum_{k=0}^{n}\Big(\frac{1}{p}\Big)^k\frac{(-n)_k}{k!}\frac{1}{(-N)_k}(-x)_k \\ &= {}_2F_1\left(\begin{matrix}-n, -x\\ -N\end{matrix}\,\middle|\, -\frac{1}{p}\right). \end{aligned}$$

Discrete orthogonal polynomials

- *Kravchuk polynomials*

  The difference equation is

$$x\Delta\nabla y(x) + \Big(-\frac{1}{q}x + \frac{Np}{q}\Big)\Delta y(x) + \lambda y(x) = 0.$$

Then,

$$\sigma_2 = 0, \quad \sigma_1 = 1, \quad \tau_1 = -\frac{1}{q} \quad \text{et} \quad \tau_0 = \frac{Np}{q}.$$

The recurrence relation for the α_i's becomes

$$\alpha_{i+1} = -\frac{(n-i)(\mu-1)}{p(i+1)(i+N)}\alpha_i = -\frac{(n-i)}{(i+1)}\frac{1}{p}\frac{1}{i+N}\alpha_i.$$

Finally we obtain the hypergeometric representation

$$\begin{aligned} K_n(x,p,N) &= \sum_{k=0}^{n}\Big(\frac{1}{p}\Big)^k\frac{(-n)_k}{k!}\frac{1}{(-N)_k}(-x)_k \\ &= {}_2F_1\left(\begin{matrix}-n,-x\\-N\end{matrix}\,\middle|\,-\frac{1}{p}\right). \end{aligned}$$

Discrete orthogonal polynomials

- *Kravchuk polynomials*
  The difference equation is

$$x\Delta\nabla y(x) + \Big(-\frac{1}{q}x + \frac{Np}{q}\Big)\Delta y(x) + \lambda y(x) = 0.$$

Then,

$$\sigma_2 = 0, \quad \sigma_1 = 1, \quad \tau_1 = -\frac{1}{q} \quad \text{et} \quad \tau_0 = \frac{Np}{q}.$$

The recurrence relation for the α_i's becomes

$$\alpha_{i+1} = -\frac{(n-i)(\mu-1)}{p(i+1)(i+N)}\alpha_i = -\frac{(n-i)}{(i+1)}\frac{1}{p}\frac{1}{i+N}\alpha_i.$$

Finally we obtain the hypergeometric representation

$$\begin{aligned} K_n(x,p,N) &= \sum_{k=0}^{n}\Big(\frac{1}{p}\Big)^k\frac{(-n)_k}{k!}\frac{1}{(-N)_k}(-x)_k \\ &= {}_2F_1\left(\begin{matrix}-n,-x\\-N\end{matrix}\,\middle|\,-\frac{1}{p}\right). \end{aligned}$$

Discrete orthogonal polynomials

- *Kravchuk polynomials*

  The difference equation is

$$x\Delta\nabla y(x) + \Big(-\frac{1}{q}x + \frac{Np}{q}\Big)\Delta y(x) + \lambda y(x) = 0.$$

Then,

$$\sigma_2 = 0, \quad \sigma_1 = 1, \quad \tau_1 = -\frac{1}{q} \quad \text{et} \quad \tau_0 = \frac{Np}{q}.$$

The recurrence relation for the α_i's becomes

$$\alpha_{i+1} = -\frac{(n-i)(\mu-1)}{p(i+1)(i+N)}\alpha_i = -\frac{(n-i)}{(i+1)}\frac{1}{p}\frac{1}{i+N}\alpha_i.$$

Finally we obtain the hypergeometric representation

$$\begin{aligned} K_n(x,p,N) &= \sum_{k=0}^{n}\Big(\frac{1}{p}\Big)^k\frac{(-n)_k}{k!}\frac{1}{(-N)_k}(-x)_k \\ &= {}_2F_1\left(\begin{matrix} -n, -x \\ -N \end{matrix} \,\Big|\, -\frac{1}{p}\right). \end{aligned}$$

Discrete orthogonal polynomials

- *Hahn polynomials*
  The difference equation is

$$(x^2+\mu x)\Delta\nabla y(x)+\tau(x)\Delta y(x)+\lambda y(x)=0$$

with

$$\tau(x)=-(2N+\mu+\nu-2)x+(N+\nu+1)(N-1).$$

$$\begin{aligned} Q_n(x,\mu,\nu,N) &= \sum_{k=0}^{n}\frac{(-n)_k(n+\nu+\mu+1)_k(-x)_k}{(\alpha+N)_k(-N)_k} \\ &= {}_3F_2\left(\begin{matrix}-n, n+\mu+\nu+1, -x\\ \alpha+N, -N\end{matrix}\,\middle|\, -\frac{1}{p}\right)\end{aligned}$$

Discrete orthogonal polynomials

- *Hahn polynomials*
  The difference equation is

$$(x^2+\mu x)\Delta\nabla y(x)+\tau(x)\Delta y(x)+\lambda y(x)=0$$

with

$$\tau(x)=-(2N+\mu+\nu-2)x+(N+\nu+1)(N-1).$$

$$\begin{aligned} Q_n(x,\mu,\nu,N) &= \sum_{k=0}^{n}\frac{(-n)_k(n+\nu+\mu+1)_k(-x)_k}{(\alpha+N)_k(-N)_k} \\ &= {}_3F_2\left(\begin{array}{c} -n, n+\mu+\nu+1, -x \\ \alpha+N, -N \end{array}\middle| -\frac{1}{p}\right) \end{aligned}$$

For tutorial

We recall that the Charlier polynomials have the hypergeometric representation

$$C_n(x,a) = {}_2F_0\left(\begin{array}{c} -n, -x \\ - \end{array} \middle| -\frac{1}{c}\right) = \sum_{k=0}^{n} \left(\frac{1}{c}\right)^k \frac{(-n)_k}{k!} (x)_{[k]}.$$

We would like to solve the following problems.

- *The inversion problem:* Find $I_m(n)$ such that

$$(x)_{[n]} = \sum_{k=0}^{n} I_k(n) C_k(x,a).$$

- *The connection formula:* Find $D_m(n)$ such that

$$C_n(x,b) = \sum_{k=0}^{n} D_k(n) C_k(x,a).$$

- *An addition formula* Find the coefficients $A_m(n)$ in the expansion

$$C_n(x+y, a+b) = \sum_{k=0}^{n} A_k(n) C_{n-k}(x,a) C_k(x,b).$$

For tutorial

We recall that the Charlier polynomials have the hypergeometric representation

$$C_n(x,a) = {}_2F_0\left(\begin{array}{c} -n, -x \\ - \end{array} \middle| -\frac{1}{c}\right) = \sum_{k=0}^{n} \left(\frac{1}{c}\right)^k \frac{(-n)_k}{k!}(x)_{[k]}.$$

We would like to solve the following problems.

- *The inversion problem:* Find $I_m(n)$ such that

$$(x)_{[n]} = \sum_{k=0}^{n} I_k(n) C_k(x,a).$$

- *The connection formula:* Find $D_m(n)$ such that

$$C_n(x,b) = \sum_{k=0}^{n} D_k(n) C_k(x,a).$$

- *An addition formula* Find the coefficients $A_m(n)$ in the expansion

$$C_n(x+y, a+b) = \sum_{k=0}^{n} A_k(n) C_{n-k}(x,a) C_k(x,b).$$

For tutorial

We recall that the Charlier polynomials have the hypergeometric representation

$$C_n(x,a) = {}_2F_0\left(\begin{array}{c} -n, -x \\ - \end{array} \middle| -\frac{1}{c} \right) = \sum_{k=0}^{n} \left(\frac{1}{c}\right)^k \frac{(-n)_k}{k!} (x)_{[k]}.$$

We would like to solve the following problems.

- *The inversion problem:* Find $I_m(n)$ such that

$$(x)_{[n]} = \sum_{k=0}^{n} I_k(n) C_k(x,a).$$

- *The connection formula:* Find $D_m(n)$ such that

$$C_n(x,b) = \sum_{k=0}^{n} D_k(n) C_k(x,a).$$

- *An addition formula* Find the coefficients $A_m(n)$ in the expansion

$$C_n(x+y, a+b) = \sum_{k=0}^{n} A_k(n) C_{n-k}(x,a) C_k(x,b).$$

Basic notations and definitions

In what follows, q is a real number such that $0 < q < 1$.

q-pocchammer, q-number, q-binomial

The q-Pochhammer is defined by

$$(a;q)_n = (1-a)(1-aq)\dots(1-aq^{n-1}), \quad (a;q)_0 = 1.$$

The so-called q-number (or q-bracket) is defined by

$$[a]_q = \frac{1-q^a}{1-q}, \quad a \in \mathbb{C}, \quad q \neq 1,$$

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_m (q;q)_{n-m}}, \quad 0 \leq m \leq n,$$

and the q-power is

$$(x \ominus y)_q^n = (x-y)(x-qy)\cdots(x-q^{n-1}y). \tag{10}$$

Basic notations and definitions

The q-hypergeometric series

The basic hypergeometric or q-hypergeometric series ${}_r\phi_s$ is defined by the series

$$ {}_r\phi_s\left(\begin{array}{c} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{array} \middle| q; z\right) := \sum_{n=0}^{\infty} \frac{(a_1, \cdots, a_r; q)_n}{(b_1, \cdots, b_s; q)_n} \left((-1)^n q^{\binom{k}{2}}\right)^{1+s-r} \frac{z^n}{(q;q)_n}, $$

where

$$ (a_1, \cdots, a_r)_n := (a_1; q)_n \cdots (a_r; q)_n, $$

Basic notations and definitions

The q-deriative

The q-derivative operator is defined by

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}, \quad x \neq 0,$$

and

$$D_q f(0) = f'(0)$$

provided that f is differentiable at $x = 0$.

q-integral

Basic notations and definitions

The q-integral

Suppose $0 < a < b$. The definite q-integral is defined as

$$\int_0^b f(x)d_qx = (1-q)b\sum_{n=0}^{\infty} q^n f(q^n b), \tag{11}$$

and

$$\int_a^b f(x)d_qx = \int_0^b f(x)d_qx - \int_0^a f(x)d_qx. \tag{12}$$

Basic notations and definitions

q-Gamma function

The q-Gamma function is defined by

$$\Gamma_q(x) := \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \quad 0 < q < 1. \tag{13}$$

Note also that the q-Gamma function satisfies the functional equation

$$\Gamma_q(x+1) = [x]_q\Gamma_q(x), \quad \text{with} \quad \Gamma_q(1) = 1.$$

Exercise. Proof the previous difference equation for Γ_q.

q-exponential

Basic notations and definitions

q-Gamma function

The q-Gamma function is defined by

$$\Gamma_q(x) := \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \quad 0<q<1. \tag{13}$$

Note also that the q-Gamma function satisfies the functional equation

$$\Gamma_q(x+1) = [x]_q\Gamma_q(x), \quad \text{with} \quad \Gamma_q(1) = 1.$$

Exercise. Proof the previous difference equation for Γ_q.

q-exponential

Basic notations and definitions

The q-exponential functions

There are two most known q-extension of the exponential function:

$$e_q(z) := \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!} = \frac{1}{((1-q)z;q)_\infty}, \quad 0 < |q| < 1, \ \ |z| < 1,$$

and

$$E_q(z) := \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{[n]_q!} z^n = (-(1-q)z;q)_\infty, \quad 0 < |q| < 1.$$

These q-analogues of the exponential function are clearly related by

$$e_q(z)E_q(-z) = 1.$$

Orthogonal polynomials of the q-discrete variable

A polynomial set $p_n(x)$ given by (1), is a family of classical q-discrete orthogonal polynomials (also known as the polynomials of the q-Hahn tableau) if it is the solution of a q-difference equation of the type

$$\sigma(x)D_qD_{q^{-1}}y(x) + \tau(x)D_qy(x) + \lambda_n y(x) = 0. \tag{14}$$

Here the polynomials $\sigma(x)$ and $\tau(x)$ are known to satisfy a Pearson type equation

$$D_q(\sigma(x)\rho(x)) = \tau(x)\rho(x),$$

where the function $\rho(x)$ is the q-discrete weight function associated to the family.

Polynomial solutions of the q-difference equation

We find the solutions having the following three represenations

- **Representation in the power basis**
  In this case we seek for the solution in the form

$$P_n(x) = \sum_{m=0}^{n} C_m(n) x^m$$

- **Representation in the q-Pochhammer basis**
  In this case we seek for the solution in the form

$$P_n(x) = \sum_{m=0}^{n} C_m(n) (x;q)_k$$

- **Representation in the q-power basis**
  In this case we seek for the solution in the form

$$P_n(x) = \sum_{m=0}^{n} C_m(n) (x \ominus 1)_q^k.$$

Polynomial solutions of the q-difference equation

We find the solutions having the following three represenations

- **Representation in the power basis**
  In this case we seek for the solution in the form

$$P_n(x) = \sum_{m=0}^{n} C_m(n) x^m$$

- **Representation in the q-Pochhammer basis**
  In this case we seek for the solution in the form

$$P_n(x) = \sum_{m=0}^{n} C_m(n) (x;q)_k$$

- **Representation in the q-power basis**
  In this case we seek for the solution in the form

$$P_n(x) = \sum_{m=0}^{n} C_m(n) (x \ominus 1)_q^k.$$

Polynomial solutions of the q-difference equation

We find the solutions having the following three represenations

- **Representation in the power basis**
  In this case we seek for the solution in the form

$$P_n(x) = \sum_{m=0}^{n} C_m(n) x^m$$

- **Representation in the q-Pochhammer basis**
  In this case we seek for the solution in the form

$$P_n(x) = \sum_{m=0}^{n} C_m(n) (x;q)_k$$

- **Representation in the q-power basis**
  In this case we seek for the solution in the form

$$P_n(x) = \sum_{m=0}^{n} C_m(n) (x \ominus 1)_q^k.$$

Polynomial solutions of the q-difference equation

Representation in the power basis

Let $P_n(x)$ be a polynomial system given by the q-differential equation (14) with $\sigma(x) = ax^2 + bx + c$, and $\tau(x) = dx + e$. Then, the power series coefficients $C_m(n)$ given by

$$P_n(x) = \sum_{m=0}^{n} C_m(n)x^m \qquad (15)$$

satisfy the recurrence equation

$$\begin{aligned}(a[m]_{\frac{1}{q}}[m-1]_q + d[m]_q - \lambda_n)C_m(n) \\ +(b[m+1]_{\frac{1}{q}}[m]_q + e[m+1]_q)C_{m+1}(n) \\ +c[m+2]_{\frac{1}{q}}[m+1]_q C_{m+2}(n) = 0, \qquad (16)\end{aligned}$$

with $C_n(n) = 1, C_{n+1}(n) = 0.$

Polynomial solutions of the q-difference equation

Representations in the power basis

In particular, if $c = 0$, then the recurrence equation

$$(a[m]_{1/q}[m-1]_q + d[m]_q - \lambda_n)C_m(n) + (b[m+1]_{1/q}[m]_q + e[m+1]_q)C_{m+1}(n), \tag{17}$$

is valid, and therefore $P_n(x)$ has the following q-hypergeometric representation up to a constant K_n:

$$P_n(x) = K_n \; {}_2\phi_1\left(\begin{array}{c} q^{-n}, \frac{a-d+dq}{a}q^{n-1} \\ \frac{b-e+eq}{b} \end{array}\middle| q; -\frac{aq}{b}x\right), \qquad ab \neq 0, \tag{18}$$

$$P_n(x) = K_n \; {}_1\phi_1\left(\begin{array}{c} q^{-n} \\ \frac{b-e+eq}{b} \end{array}\middle| q; \frac{d(1-q)q^n}{b}x\right) \qquad a = 0,\ b \neq 0, \tag{19}$$

$$P_n(x) = K_n \; {}_1\phi_0\left(\begin{array}{c} q^{-n} \\ - \end{array}\middle| q; -\frac{dq^n}{e}x\right), \qquad a = b = 0. \tag{20}$$

Polynomial solutions of the q-difference equation

Representations in the power basis

Let $P_n(x)$ be a polynomial system given by the q-differential equation (14) with $\sigma(x) = ax^2 + bx + c$, and $\tau(x) = dx + e$. Then, the power series coefficients $C_m(n)$ given by

$$P_n(x) = \sum_{m=0}^{n} C_m(n)(x;q)_m \tag{21}$$

satisfy the recurrence equation

$$\begin{aligned} & q^n \left(q^{m+2} - 1\right) \left(q^{m+1} - 1\right) \left(a + q^{m+1} b + c q^{2\,m+2}\right) C_{m+2}(n) \\ & - \left(q^{m+1} - 1\right) q \Big(- q^{n+1} a - a q^n + q^{n+2\,m+1} b - q^{m+1+n} b + q^{2\,m+2+n} e \\ & - q^{n+2\,m+1} e + q^{m+2\,n} a + q^{m+2\,n+1} d - q^{m+2\,n} d + q^{m+1} a \Big) C_{m+1}(n) \\ & - \left(-q^m + q^n\right) \left(q^{n+m} a + q^{m+1+n} d - q^{n+m} d - a q\right) q^2 C_m(n) = 0, \end{aligned} \tag{22}$$

where $m = -2, -1, 0, \cdots, n$ and $C_m(n) = 0$ outside the set of (n, m) such that $0 \le m \le n$,with $C_n(n) = 1, C_{n+1}(n) = 0$.

Polynomial solutions of the q-difference equation

Representations in the power basis

The following representations of monic orthogonal polynomials of the q-Hahn class follows:

- the Big q-Jacobi polynomials

$$\tilde{P}_n(x,\alpha,\beta,\gamma;q) = \frac{(\alpha q;q)_n(\gamma q;q)_n}{(\alpha\beta q^{n+1};q)_n}\, {}_3\phi_2\left(\begin{array}{c} q^{-n}, \alpha\beta q^{n+1}, x \\ \alpha q, \gamma q \end{array} \middle| q;q \right),$$

- the q-Hahn polynomials

$$\tilde{Q}_n(x,\alpha,\beta,N,q) = \frac{(\alpha q;q)_n(q^{-N};q)_n}{(\alpha\beta q^{n+1};q)_n}\, {}_3\phi_2\left(\begin{array}{c} q^{-n}, \alpha\beta q^{n+1}, x \\ \alpha q, q^{-N} \end{array} \middle| q;q \right)$$

- the Big q-Laguerre polynomials

$$tildeP_n(x,\alpha,\beta,q) = (\alpha q;q)_n(\beta q;q)_{n3}\phi_2\left(\begin{array}{c} q^{-n}, 0, x \\ \alpha q, \beta q \end{array} \middle| q;q \right)$$

Polynomial solutions of the q-difference equation

- the Little q-Jacobi polynomials

$$\tilde{p}_n(x;\alpha,\beta|q) = \frac{(-1)^n q^{\binom{n}{2}}(\alpha q;q)_n}{(\alpha\beta q^{n+1};q)_n}\,{}_2\phi_1\left(\begin{array}{c} q^{-n},\alpha\beta q^{n+1} \\ \alpha q \end{array}\middle| q;qx\right)$$

- the Alternative q-Charlier polynomials

$$\tilde{K}_n(x,\alpha,q) = \frac{(-1)^n q^{\frac{n(n-1)}{2}}}{(-\alpha q^n;q)_n}\,{}_2\phi_1\left(\begin{array}{c} q^{-n},-\alpha q^n \\ 0 \end{array}\middle| q;qx\right)$$

- the Little q-Laguerre/Wall polynomials

$$\tilde{p}_n(x,\alpha|q) = (-1)^n q^{\binom{n}{2}}(\alpha q;q)_{n}\,{}_2\phi_1\left(\begin{array}{c} q^{-n},0 \\ \alpha q \end{array}\middle| q,qx\right)$$

Polynomial solutions of the q-difference equation

- the q-Meixner polynomials

$$\tilde{M}_n(x,\beta,\gamma;q) = (-\gamma)^n q^{n^2} (\beta q;q)_n {}_2\phi_1\left(\begin{array}{c} q^{-n}, x \\ \beta q \end{array} \middle| q; -\frac{q^{n+1}}{\gamma} \right)$$

- the q-Charlier polynomials

$$\tilde{C}_n(x,\alpha,q) = (-\alpha)^n q^{-n^2} {}_2\phi_1\left(\begin{array}{c} q^{-n}, x \\ 0 \end{array} \middle| q; -\frac{q^{n+1}}{\alpha} \right)$$

- the q-Laguerre polynomials

$$\tilde{L}_n^{(\alpha)}(x;q) = \frac{(-1)^n (q^{\alpha+1} : q)_n}{q^{n(n+\alpha)}} {}_1\phi_1\left(\begin{array}{c} q^{-n} \\ q^{\alpha+1} \end{array} \middle| q; -xq^{\alpha+n+1} \right)$$

Polynomial solutions of the q-difference equation

and

- the Stieltjes-Wigert polynomials

$$\tilde{S}_n(x;q) = (-1)^n q^{-n^2} {}_1\phi_1\left(\begin{array}{c} q^{-n} \\ 0 \end{array} \middle| q; -xq^{n+1} \right)$$

- the Al Salam-Carlitz II polynomials

$$\tilde{V}_n^{(\alpha)}(x,q) = (-\alpha)^n q^{-\binom{n}{2}} {}_2\phi_0\left(\begin{array}{c} q^{-n}, x \\ - \end{array} \middle| q; \frac{q^n}{\alpha} \right)$$

- the Discrete q-Hermite II polynomials

$$\tilde{H}_n(x,q) = q^{-\frac{n(n-1)}{2}} {}_2\phi_0\left(\begin{array}{c} q^{-n}, x \\ - \end{array} \middle| q, -q^n \right)$$

Polynomial solutions of the q-difference equation

Representations in the q-power basis

Let $P_n(x)$ be a polynomial system given by the q-differential equation (14) with $\sigma(x) = (x-1)(x-a)$, and $\tau(x) = dx + e$. Then, the power series coefficients $C_m(n)$ given by

$$P_n(x) = \sum_{m=0}^{n} C_m(n)(x \ominus 1)_q^m, \tag{23}$$

satisfy the recurrence relation

$$\begin{aligned}\left(q^{m+1}-1\right)\left(q^{2m}-q^{m}-aq^{m}+a+dq^{2m+1}-dq^{2m}+eq^{m+1}-eq^{m}\right)C_{m+1}(n)\\ +\left(-q^{m+1}+q^{2m}+q-q^{m}+dq^{2m+1}-dq^{2m}-dq^{m+1}+dq^{m}+\lambda_n\,q^{m+2}\right.\\ \left.-2\,\lambda_n\,q^{m+1}+\lambda_n\,q^{m}\right)C_m(n)=0.\end{aligned} \tag{24}$$

From this we get the following q-hypergeometric representation

- the Al-Salam-Carlitz I polynomials

$$U_n^{(a)}(x;q) = (-a)^n q^{\binom{n}{2}} {}_2\phi_0\left(\begin{array}{c} q^{-n}, x^{-1} \\ 0 \end{array} \middle| \frac{qx}{a} \right)$$

- the discrete q-Hermite I polynomials

$$h_n(x;q) = U_n^{(-1)}(x;q) = q^{\binom{n}{2}} {}_2\phi_0\left(\begin{array}{c} q^{-n}, x^{-1} \\ 0 \end{array} \middle| -qx \right).$$

Thank you for your attention!

We move to some tutorials...

Tutorial 1

The generating function for the Charlier polynomials is given by

$$e^t\left(1-\frac{t}{a}\right)^x=\sum_{n=0}^{\infty}C_n(x;a)\frac{t^n}{n!}.$$

1. Prove the inversion formula

$$(x)_{[n]}=\sum_{k=0}^{n}(-1)^k\binom{n}{k}a^nC_k(x;a).$$

2. Prove the connection formula

$$C_n(x;b)=\frac{1}{b^n}\sum_{k=0}^{n}\binom{n}{k}(b-a)^{n-k}a^kC_k(x;a).$$

3. Prove the addition formula

$$C_n(x+y;a+b)=\sum_{k=0}^{n}\binom{n}{k}\frac{a^kb^{n-k}}{(a+b)^n}C_k(x;a)C_{n-k}(y;b).$$

Tutorial 2

1. Prove that $D_q^k x^n = \dfrac{[n]_q!}{[n-k]_q!} x^k$ and $D_q(x \ominus a)_q^n = \dfrac{[n]_q!}{[n-k]_q!}(x \ominus a)_q^{n-k}$.
2. For a polynomial $f(x)$ of degree N, prove the q-Taylor expansion

$$f(x) = \sum_{k=0}^{N} \frac{(D_q^k f)(a)}{[k]_q}(x \ominus a)_q^k.$$

3. Deduce that for a polynomial $f(x)$ of degree N,

$$f(x) = \sum_{k=0}^{N} \frac{(D_q^k f)(0)}{[k]_q} x^k.$$

4. Deduce the connection formula

$$(x \ominus a)_q^n = \sum_{k=0}^{n} q^{\binom{n-k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q (-a)^{n-k} x^k$$

5. Prove the formula $e_q(x)E_q(-x) = 1$.

Tutorial 3

Note that the Al-Salam Carlitz-I polynomials are generated by the q-exponential generating function

$$\frac{e_q(xt)}{e_q(t)e_q(at)} = \sum_{n=0}^{\infty} U_n^{(a)}(x;q)\frac{t^n}{[n]_q!}, \tag{25}$$

1. Prove the inversion formula

$$(x \ominus 1)_q^n = \sum_{k=0}^{n} a^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_q U_k^{(a)}(x;q).$$

2. Prove the connection formula

$$U_n^{(b)}(x;q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (a \ominus b)_q^k U_k^{(a)}(x;q).$$

3. Prove the q-addition formula

$$U_n^{(a)}(x \oplus_q y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q y^{n-k} U_k^{(a)}(x;q).$$

Tutorial 3

where the notation $(a \oplus_q b)^n$ stands for the Ward q-addition and is defined by

$$(a \oplus_q b)^n := \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q a^k b^{n-k}, \quad n = 0, 1, 3, \ldots .$$

Thank you for your attention!