Three systems of orthogonal polynomials and L^2-boundedness of two associated operators

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Overview

This talk is based on our recent article,

> Musonda, J., Kaijser, S. (2018), *Three systems of orthogonal polynomials and L^2-boundedness of two associated operators*, J. Math. Anal. Appl. **459** 464–475,

which is in two parts.

1. Presents three systems of orthogonal polynomials belonging to the class of Meixner–Pollaczek polynomials, and some connections between them.

2. Investigates boundedness properties of two singular integral operators of convolution type in the Hilbert spaces related to the three systems. Orthogonal polynomials are used to prove boundedness in the weighted spaces.

Basic objects of study

Three systems of orthogonal polynomials and three operators.

σ	τ	ρ
$\sigma_0 = 1$	$\tau_0 = 1$	$\rho_0 = 1$
$\sigma_1 = x$	$\tau_1 = x$	$\rho_1 = x$
$\sigma_2 = x^2$	$\tau_2 = x^2 - 1$	$\rho_2 = x^2 - 2$
$\sigma_3 = x^3 - 2x$	$\tau_3 = x^3 - 5x$	$\rho_3 = x^3 - 8x$
$\sigma_4 = x^4 - 8x^2$	$\tau_4 = x^4 - 14x^2 + 9$	$\rho_4 = x^4 - 20x^2 + 24$
$\vdots$	$\vdots$	$\vdots$

$$Rf(x) = \frac{f(x+i) + f(x-i)}{2} \qquad \sigma_n \xrightarrow{R} \tau_n \xrightarrow{R} \rho_n$$

$$Jf(x) = \frac{f(x+i) - f(x-i)}{2i} \qquad \sigma_n \xrightarrow{J} n\tau_{n-1} \xrightarrow{J} n(n-1)\rho_{n-2}$$

$$Qf(x) = xf(x) \qquad \sigma_{n+1} \xleftarrow{Q} \rho_n$$

Background of the research problem

In his article,

Sten Kaijser (1999), *Några nya ortogonala polynom*, Normat **47** 156–165,

Sten Kaijser presented two systems of orthogonal polynomials belonging to the class of Meixner–Pollaczek polynomials

- $\{\tau_n\}$ – orthogonal on the real line $\mathbb{R}$ with respect to

$$\omega_1(x) = \frac{1}{2\cosh \frac{\pi}{2}x}. \tag{1}$$

- $\{\sigma_n\}$ – orthogonal in the strip $\mathbb{S}$ with respect to ω_1.

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- $\{\tau_n\}$ – orthogonal on the real line $\mathbb{R}$ with respect to

$$\omega_1(x) = \frac{1}{2\cosh\frac{\pi}{2}x}. \qquad (1)$$

- $\{\sigma_n\}$ – orthogonal in the strip $\mathbb{S}$ with respect to ω_1.

These polynomials were studied in a series of papers by Araaya.

[1] T.K. Araaya (2004), *Linearization and connection problems for the symmetric Meixner–Pollaczek polynomials*, Int. J. Pure Appl. Math. **17** 409–422

[2] Tsehaye Araaya (2004), *The Meixner–Pollaczek polynomials and a system of orthogonal polynomials in a strip*, J. Comput. Appl. Math. **170** 241–254

[3] Tsehaye Araaya (2005), *The symmetric Meixne–Pollaczek polynomials with real parameter*, J. Math. Anal. Appl. 305 411–423

[4] Tsehaye Araaya (2003), *Umbral Calculus and the Meixner–Pollaczek Polynomials*, Uppsala Dissertations in Mathematics, Department of Mathematics, Uppsala University, ISBN91-506-1681-1,

Motivation

- The Sten–Araaya systems are connected by two operators,

$$Rf(x) = \frac{f(x+i) + f(x-i)}{2}, \tag{2}$$

$$Jf(x) = \frac{f(x+i) - f(x-i)}{2i}. \tag{3}$$

- Later on, it was observed that

$$\tau_n \xrightarrow{\quad R \quad} \rho_n \qquad \sigma_{n+1} \xleftarrow{\quad Q \quad} \rho_n \tag{4}$$

where $Qf(x) = xf(x)$.

- Inspired by Lars Holst,

Lars Holst (2013), *Probabilistic proofs of Euler identities*, J. Appl. Probab,

$\{\rho_n\}$ – orthogonal on the real line $\mathbb{R}$ with respect to

$$\omega_2(x) = (\omega_1 * \omega_1)(x) = \frac{x}{2\sinh\frac{\pi}{2}x}. \tag{5}$$

Orthogonal polynomials on the real line $\mathbb{R}$

1. A function ω on $\mathbb{R}$ is called a polynomially bounded weight function if

$$\omega \geq 0, \quad 0 < \int_{\mathbb{R}} \omega(x) dx < \infty \quad \text{and} \quad 0 < \int_{\mathbb{R}} |x|^n \omega(x) dx < \infty. \quad (6)$$

2. For real polynomials f and g on the real line $\mathbb{R}$, we can define an inner product and the corresponding norm:

$$(f, g)_{\mathbb{R},\omega} = \int_{\mathbb{R}} f(x)\overline{g(x)}\omega(x)\, dx \quad \text{and} \quad ||f||_{\mathbb{R},\omega} = \sqrt{(f, f)_{\mathbb{R},\omega}} \quad (7)$$

3. A system $\{p_n\}_{n=0}^{\infty}$ of such polynomials, where every polynomial p_n has degree n, is called orthogonal if for $n \neq m$,

$$(p_n, p_m)_{\mathbb{R},\omega} = 0, \quad (8)$$

and it is called orthonormal if $(p_n, p_n)_{\mathbb{R},\omega} = 1$ for all n.

4. Construction: Gram–Schmidt procedure applied to $1, x, x^2, x^3, \cdots$

Orthogonal polynomials in the strip $\mathbb{S}$

Let ω be a weight. For analytic polynomials f and g in the strip

$$\mathbb{S} = \{z \in \mathbb{C} : |\text{Im}\, z| < 1\}, \tag{9}$$

we can define an inner product and the corresponding norm:

$$(f,g)_{\mathbb{S},\omega} = \int_{\mathbb{R}} \frac{f(x+i)\overline{g(x+i)} + f(x-i)\overline{g(x-i)}}{2}\omega(x)dx, \tag{10}$$

$$||f||_{\mathbb{S},\omega} = \sqrt{(f,f)_{\mathbb{S},\omega}}, \tag{11}$$

respectively. A system $\{p_n\}_{n=0}^{\infty}$ of such polynomials, where every polynomial p_n has degree n, is called orthogonal if for $n \neq m$,

$$(p_n, p_m)_{\mathbb{S},\omega} = 0. \tag{12}$$

Such an orthogonal system is called orthonormal if for all n,

$$(p_n, p_n)_{\mathbb{S},\omega} = 1. \tag{13}$$

Construction: Gram–Schmidt procedure applied to $1, x, x^2, x^3, \cdots$

Three-term recurrence relation

Proposition

For any given weight function ω, there exists a unique system $\{p_n\}_{n=0}^{\infty}$ of monic orthogonal polynomials. More precisely, we can construct the monic orthogonal polynomials as follows:

$$
\begin{aligned}
p_{-1}(x) &= 0, \\
p_0(x) &= 1, \\
p_{n+1}(x) &= xp_n(x) - a_n p_n(x) - b_n p_{n-1}(x),
\end{aligned}
$$

where $a_n = (xp_n, p_n)_\omega / \|p_n\|_\omega^2$ and $b_n = (xp_n, p_{n-1})_\omega / \|p_{n-1}\|_\omega^2$.

1. This is simply the **Gram–Schmidt** procedure applied to the sequence $\{x^n\}_{n=0}^{\infty}$ with respect to the inner product $(p_n, p_m)_\omega$.

2. The converse of this theorem is known as **Favard's theorem**.

Three-term recurrence relation: Our case

The weight

$$\omega_1(x) = 1/(2\cosh\frac{\pi}{2}x) \tag{14}$$

is even. Thus, the three-term recurrence relation reduces to

$$p_{-1}(x) = 0, \quad p_0(x) = 1, \quad p_{n+1}(x) = xp_n(x) - b_n p_{n-1}(x). \tag{15}$$

In particular, for the systems $\{\sigma_n\}$, $\{\tau_n\}$, $\{\rho_n\}$, we have

$$b_n = n(n-1),$$
$$b_n = n(n+0),$$
$$b_n = n(n+1),$$

respectively.

- Denoted by $p_n^{(\lambda)}(x;\phi)$ where $\lambda > 0$ and $0 < \phi < \pi$.
- Discovered by Meixner and later studied by Pollaczek.
- Defined by the recurrence relation

$$
\begin{aligned}
p_{-1}^{(\lambda)}(x;\phi) &= 0,\\
p_{0}^{(\lambda)}(x;\phi) &= 1,\\
(n+1)p_{n+1}^{(\lambda)}(x;\phi) &= 2\big(x\sin\phi + (n+\lambda)\cos\phi\big)p_n^{(\lambda)}(x;\phi)\\
&\qquad - (n+2\lambda-1)p_{n-1}^{(\lambda)}(x;\phi).
\end{aligned}
$$

- The generating function, $G_\lambda(x,s) = \sum_{n=0}^{\infty} p_n^{(\lambda)}(x;\phi)s^n$, is

$$
G_\lambda(x,s) = \left(1 - se^{i\phi}\right)^{-\lambda+ix}\left(1 - se^{-i\phi}\right)^{-\lambda-ix}. \tag{16}
$$

- Symmetric Meixner–Pollaczek polynomials, $p_n^{(\lambda)}(x/2;\pi/2)$

The Symmetric Meixner–Pollaczek polynomials

- Denoted by $p_n = p_n^{(\lambda)}(x/2; \pi/2)$, where $\lambda > 0$.
- These polynomials $\{p_n\}_{n=0}^{\infty}$ are defined by the recurrence

$$\begin{aligned} p_{-1}(x) &= 0 \\ p_0(x) &= 1 \\ p_{n+1}(x) &= xp_n(x) - n(n-1+2\lambda)p_{n-1}(x). \end{aligned}$$

Thus, the systems $\{\sigma_n\}$, $\{\tau_n\}$, $\{\rho_n\}$ are respectively the cases

$$\lambda \to 0, \quad \lambda = 1/2, \quad \lambda = 1 \tag{17}$$

- The generating function, $G_\tau(x, s) = \sum_{n=0}^{\infty} \frac{\tau_n(x)}{n!} s^n$, is

$$G_\tau(x, s) = \frac{e^{x \arctan s}}{(1+s^2)^\lambda}. \tag{18}$$

- The weight is given by

$$\omega_\lambda(x) = \frac{\left|\Gamma(\lambda + ix/2)\right|^2}{2\pi}. \tag{19}$$

The ρ-system

The following theorem gives a description of the ρ-system together with the basic properties of the system.

Theorem

Let the system $\{\rho_n\}_{n=0}^{\infty}$ be given by the recurrence relation

$$\rho_{-1} = 0, \ \rho_0 = 1 \text{ and } \rho_{n+1}(x) = x\rho_n(x) - n(n+1)\rho_{n-1}(x). \quad (20)$$

Then the following hold:

(a) *The function ρ_n is a monic polynomial of degree n for $n \geq 0$.*

(b) *The generating function, $G_\rho(x,s) = \sum_{n=0}^{\infty} \rho_n(x)\frac{s^n}{n!}$, is given by the function*

$$G_\rho(x,s) = \frac{e^{x \arctan s}}{1+s^2}. \quad (21)$$

(c) *The sequence of polynomials $\{\frac{\rho_n}{n!}\}_{n=0}^{\infty}$ is an orthogonal basis for the Hilbert space $L^2(\omega_2)$ with norms $\sqrt{n+1}$.*

Connections between the three systems

The three systems connected by three operators.

σ	τ	ρ
$\sigma_0 = 1$	$\tau_0 = 1$	$\rho_0 = 1$
$\sigma_1 = x$	$\tau_1 = x$	$\rho_1 = x$
$\sigma_2 = x^2$	$\tau_2 = x^2 - 1$	$\rho_2 = x^2 - 2$
$\sigma_3 = x^3 - 2x$	$\tau_3 = x^3 - 5x$	$\rho_3 = x^3 - 8x$
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$\vdots$	$\vdots$	$\vdots$

$$Rf(x) = \frac{f(x+i) + f(x-i)}{2} \qquad \sigma_n \xrightarrow{R} \tau_n \xrightarrow{R} \rho_n$$

$$Jf(x) = \frac{f(x+i) - f(x-i)}{2i} \qquad \sigma_n \xrightarrow{J} n\tau_{n-1} \xrightarrow{J} n(n-1)\rho_{n-2}$$

$$Qf(x) = xf(x) \qquad \sigma_{n+1} \xleftarrow{Q} \rho_n$$

Connections between the three systems

- Let the operators $K,\ L,\ M,\ N,\ O$ and P be defined as follows: $K = RRQ$, $L = QRR$, $M = RQR$, $N = RQJ$, $O = QJR$ and $P = RJQ$. Then the following relations hold:

$$K^n(\rho_0) = \rho_n, \tag{22}$$
$$L^n(\sigma_0) = \sigma_n, \tag{23}$$
$$M^n(\tau_0) = \tau_n, \tag{24}$$
$$N(\tau_n) = n\tau_n, \tag{25}$$
$$O(\sigma_n) = n\sigma_n, \tag{26}$$
$$P(\rho_n) = (n+1)\rho_n. \tag{27}$$

- Denote the polynomials $\frac{\sigma_n}{n!}, \frac{\tau_n}{n!}, \frac{\rho_n}{n!}$ by $\tilde{\sigma}_n, \tilde{\tau}_n, \tilde{\rho}_n$ respectively. The following relations hold:

$$\tilde{\sigma}_n(x \pm i) = \tilde{\tau}_n(x) \pm i\tilde{\tau}_{n-1}(x), \tag{28}$$
$$\tilde{\tau}_n(x \pm i) = \tilde{\rho}_n(x) \pm i\tilde{\rho}_{n-1}(x). \tag{29}$$

The operators B and S

- Besides the operators R, J and Q, the operators

$$B = R^{-1} \tag{30}$$

$$S = JR^{-1} \tag{31}$$

have interesting properties with respect to the three systems.

- Both operators can be represented as convolution operators:

$$Bf(z) = \int_{-\infty}^{\infty} \frac{f(t)dt}{2\cosh \frac{\pi}{2}(z-t)}, \tag{32}$$

$$Sf(x) = \lim_{\varepsilon \to 0^+} \int_{|x-t|>\varepsilon} \frac{f(t)dt}{2\sinh \frac{\pi}{2}(x-t)}. \tag{33}$$

- For our consideration:

B : functions on $\mathbb{R}$ $\rightarrow$ functions on $\mathbb{S} = \{z \in \mathbb{C} : |\mathrm{Im}\, z| < 1\}$

S : functions on $\mathbb{R}$ $\rightarrow$ functions on $\mathbb{R}$

Some notation

$L^2(\omega)$ denotes the space of measurable functions on $\mathbb{R}$ with

$$||f||^2_{L^2(\omega)} = \int_{-\infty}^{\infty} \left|f(x)\right|^2 \omega(x)\, dx < \infty. \tag{34}$$

$H^2(\omega)$ denotes the space of analytic functions on $\mathbb{S}$ with

$$||f||^2_{H^2(\omega)} = \max \int_{-\infty}^{\infty} |f(x \pm i)|^2 \omega(x)\, dx < \infty. \tag{35}$$

Furthermore, $L^2(\mathbb{R}) = L^2(1)$ and $H^2(\mathbb{S}) = H^2(1)$.

L^2-boundedness of B and S

In the following results, orthogonal polynomials are used to prove boundedness of the operators B and S in the weighted spaces.

Theorem

The operator S is linear and bounded with norm 1 on

(a) $L^2(\mathbb{R})$,
(b) $L^2(\omega_1)$,
(c) $L^2(\omega_2)$.

Theorem

The operator B is linear and bounded with norm ≤ 2 from

(a) $L^2(\mathbb{R})$ *to* $H^2(\mathbb{S})$,
(b) $L^2(\omega_1)$ *to* $H^2(\omega_1)$,
(c) $L^2(\omega_2)$ *to* $H^2(\omega_2)$.

Reordering and orthogonal polynomials

The operators R, J and Q satisfy the commutation relations

$$RQ - QR = -J, \tag{36}$$

$$JQ - QJ = R, \tag{37}$$

$$RJ - JR = 0. \tag{38}$$

An arbitrary word (monomial) ω in R, J and Q can be written as

$$\omega = R^{r_1} J^{s_1} Q^{t_1} R^{r_2} J^{s_2} Q^{t_2} \ldots R^{r_n} J^{s_n} Q^{t_n} \equiv \prod_{j=1}^{n} R^{r_j} J^{s_j} Q^{t_j}. \tag{39}$$

The main goal is to move all Qs to the left. Thus, ω is called normal ordered if all powers of Q stand to the left,

$$\omega = \sum_{r,s,t \in \mathbb{N}_0} A_{rst}(\omega) Q^r R^s J^t. \tag{40}$$

The coefficients $A_{rst}(\omega)$ are called normal ordering coefficients of ω, and sometimes, have a connection to orthogonal polynomials.

More generally

If S, T and Q are elements of an algebra satisfying the relations

$$SQ = \sigma(Q)S \quad \text{and} \quad TQ = \tau(Q)T, \tag{41}$$

where σ and τ are polynomials, then for all positive integers J, k, l,

$$S^j T^k Q^l = \Big((\tau^{\circ k} \circ \sigma^{\circ j})(Q)\Big)^l S^j T^k, \tag{42}$$

$$\Big(S^j T^k Q^l\Big)^r = \left(\prod_{t=1}^{r} \Big((\tau^{\circ k} \circ \sigma^{\circ j})^{\circ t}(Q)\Big)^{l_t}\right)(S^j T^k)^r, \tag{43}$$

and for all positive integers j_t, k_t, l_t and r, where $t = 1, \ldots, r$,

$$\prod_{t=1}^{r} S^{j_t} T^{k_t} Q^{l_t} = \left(\prod_{t=1}^{r} \Big((\tau^{\circ k_t} \circ \sigma^{\circ j_t} \circ \cdots \circ \tau^{\circ k_1} \circ \sigma^{\circ j_1})(Q)\Big)^{l_t}\right) \prod_{t=1}^{r} S^{j_t} T^{k_t},$$

where $\circ$ denotes composition of functions, and $\sigma^{\circ k}$ the k-fold composition of a function σ with itself.

Example: $\sigma_j(x) = c_j x$

Let c_σ and c_τ be complex numbers. If S, T and Q are elements of an algebra satisfying the relations

$$SQ = c_\sigma QS \quad \text{and} \quad TQ = c_\tau QT, \tag{44}$$

then for all positive integers j, k, l and r,

$$S^j T^k Q^l = c_\sigma^{jl} c_\tau^{kl} Q^l S^j T^k, \tag{45}$$

$$(S^j T^k Q^l)^r = c_\sigma^{jlr(r+1)/2} c_\tau^{klr(r+1)/2} Q^{lr} (S^j T^k)^r, \tag{46}$$

and for all positive integers j_t, k_t, l_t and r, where $t = 1, \ldots, r$,

$$\prod_{t=1}^{r} S^{j_t} T^{k_t} Q^{l_t} = \left(\prod_{n=1}^{r} c_\sigma^{j_n \sum_{t=n}^{r} l_t} c_\tau^{k_n \sum_{t=n}^{r} l_t} \right) Q^{\sum_{t=1}^{r} l_t} \prod_{t=1}^{r} S^{j_t} T^{k_t}. \tag{47}$$

Connection between S,T,Q- and R,J,Q-relations

Given the relations (where σ and τ are polynomials)

$$\begin{aligned} SQ &= \sigma(Q)S, \\ TQ &= \tau(Q)T, \end{aligned} \tag{48}$$

writing $R = (dS - bT)/(ad - bc)$ and $J = (aT - cS)/(ad - bc)$, where a, b, c and d are complex numbers with $ad \neq bc$, yields

$$\begin{aligned} RQ &= \frac{ad\sigma(Q) - bc\tau(Q)}{ad - bc}R + \frac{bd\sigma(Q) - bd\tau(Q)}{ad - bc}J, \\ JQ &= \frac{ad\tau(Q) - bc\sigma(Q)}{ad - bc}J + \frac{ac\tau(Q) - ac\sigma(Q)}{ad - bc}R. \end{aligned} \tag{49}$$

Observe that relations (48) are recovered from (49) for $b = c = 0$.
Observe that relations (49) reduce to relations

$$\begin{aligned} RQ - QR &= -J \\ JQ - QJ &= R \end{aligned} \tag{50}$$

for $\sigma(x) = x + i$, $\tau(x) = x - i$, $a = c = 1$, $b = i$ and $d = -i$.

An operator representation

A concrete representation of relations (48) is given by $S \mapsto \alpha_\sigma$, $T \mapsto \alpha_\tau$ and $Q \mapsto Q_y$, where for any polynomial f,

$$\begin{aligned} \alpha_\sigma(f)(y) &= f(\sigma(y)), \\ \alpha_\tau(f)(y) &= f(\tau(y)), \\ Q_y(f)(y) &= yf(y). \end{aligned} \tag{51}$$

Let $a, b, c, d \in \mathbb{C}$ with $ad \neq bc$. Then the operators

$$\begin{aligned} R_{\sigma,\tau}(f)(y) &= \frac{adf\left(\sigma(y)\right) - bcf\left(\tau(y)\right)}{ad - bc}, \\ J_{\sigma,\tau}(f)(y) &= \frac{acf\left(\tau(y)\right) - acf\left(\sigma(y)\right)}{ad - bc}, \\ Q_y(f)(y) &= yf(y), \end{aligned} \tag{52}$$

acting on $\mathbb{C}[y]$ gives a representation of relations (49).
For $\sigma(x) = x + i$, $\tau(x) = x - i$, $a = c = 1$, $b = i$ and $d = -i$, this reduces to the original operators R and J, which, as already mentioned, satisfy relations (50).

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Thank you