Some systems of multivariate orthogonal polynomials

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Example: Hermite polynomials

1. Hermite polynomials were defined by Pierre-Simon Laplace in 1810 —in a different form—.
2. They were studied in detail by Pafnuty Chebyshev in 1859. Chebyshev's work was overlooked and they were named later after Charles Hermite.
3. Hermite wrote on the polynomials in 1864 describing them as new. They were not new!
4. In later 1865 papers Hermite was the first to define the multidimensional polynomials.

$$\int_{-\infty}^{\infty} H_n(x)H_m(x)e^{-x^2/2}dx = d_n^2\delta_{n,m} = \begin{cases} d_n^2, & n = m, \\ 0, & n \neq m. \end{cases}$$

Rodrigues' representation $H_n(x) = \frac{(-1)^n}{e^{-x^2}} \frac{d^n}{dx^n} e^{-x^2}.$

Recurrence relation $H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x).$

Explicit representation $H_n(x) = n! \sum_{m=0}^{n/2} \frac{(-1)^m}{m!(n-2m)!}(2x)^{n-2m}.$

Generating function $\exp(2xt - t^2) = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}.$

Differential equation $y''(x) - 2xy'(x) + ny(x) = 0.$

Similarly, for the other families, they were introduced and later obtained some relations or generalized.

Characterizations

1. Three-term recurrence relation
2. Rodrigues formula
3. Hahn's characterization
4. Tricomi's characterization
5. Bochner's characterization

...

So that, we have a general framework. To me it is important to notice that all of the known families are solution of a second order linear differential equation of hypergeometric type

$$\sigma(x)y''(x) + \tau(x)y'(x) + \lambda y(x) = 0$$

where σ and τ are polynomials degree at most two and one, respectively, and λ is a constant.

Rodrigues formula

O. Rodrigues (1816) stated that for Legendre polynomials

$$p_n(x) = \frac{1}{2^n n! 1} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \left[(x^2 - 1)^n 1 \right]$$

In general (Tricomi, 1955),

$$p_n(x) = \frac{A_n}{\varrho(x)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} [\varrho(x)\sigma^n(x)]$$

where $\varrho(x)$ is the symmetrization factor of the differential operator i.e.

$$(\sigma(x)\varrho(x))' = \varrho(x)\tau(x),$$

and A_n is a normalizing constant.

Let us have a look to a very classical paper of W.A. Al-Salam.

1. Of course, not all the weights are here. We are imposing some conditions —which are equivalent!— Let me recall the lecture given by Prof. Marcellán on tuesday about semiclassical orthogonal polynomials, or even more, who are the orthogonal polynomials with respect to the skew normal distribution? (out of any known scheme in our domain but very useful in statistics)
2. In particular, I am interested in the second-order linear differential operator of hypergeometric type (cfr. also, the lecture of Prof. David Gómez-Ullate on wednesday).
3. Let me insist that in the univariate case we are using the word "classical" for many different equivalent concepts.

Several variables: Historical development

Already mentioned, in 1865 papers Hermite was the first to introduce multivariable orthogonal polynomials with respect to the multidimensional gaussian.
Basic reference:

P. Appell, J. Kampé de Fériet, Fonctions Hypergéométriques et Hypersphériques. Polynômes d'Hermite, Gauthier-Villars, Paris, 1926,

which contains a number of properties about generalizations of Hermite polynomials to several variables. It is a very classical book (almost 100 years ago) and it is not easy to get a copy. Thus, e.g. for $n = 2$ they analized orthogonal polynomials with domain exactly $\mathbf{R}^2$, with orthogonality weight function defined as the bivariate gaussian.

Orthogonal polynomials on a triangular region were introduced by Proriol:

J. Proriol. *Sur une famille de polynômes à deux variables orthogonaux dans un triangle* C.R. Acad. Sci. Paris **245**(5), 2459–2461, (1957).

They were applied to the problem of solving the Schrödinger equation for the Helium atom.

G. Munschy. *Résolution de l'équation de Schrödinger des atomes à deux électrons III*. J. Phys. Radium **8**(18), 552–558 (1957).

G. Munschy and P. Pluvininage. *Résolution de l'équation de Schrödinger des atomes à deux électrons II*. J. Phys. Radium **8**(18), 157–160 (1957).

The same class was independently obtained by Karlin and McGregor

S. Karlin and J. McGregor. *On some stochastic models in genetics*. In "Stochastic models in medicine and biology", J. Gurland editor, University of Wisconsin Press, Madison, 1964 245–271.

in view of applications to genetics, as indicated by Koornwinder, first systematic study on classical orthogonal polynomials of two variables.

T. Koornwinder. *Two–variable analogues of the classical orthogonal polynomials*. In "Theory and Application of Special Functions", R. Askey ed., Proc. Adv. Semin., The University of Wisconsin–Madison, Academic Press 1975, 435–495.

More concretely, Koornwinder gave a general method of generating orthogonal polynomials of two variables from orthogonal polynomials of one variable. This method was also discussed by Dunkl and Xu in their excellent book (available in the room).
Ch. F. Dunkl and Y. Xu *Orthogonal Polynomials of Several variables* . Encyclopedia of Mathematics and Its Applications **81**. (CUP, 2001).

Krall and Sheffer and independently Engelis considered some two–dimensional analogues of classical orthogonal polynomials which are solutions of linear partial differential equations of the second order.

H.L. Krall and I.M. Sheffer. *Orthogonal polynomials in two variables*. Ann. Mat. Pura Appl. **4**(76), 325–376 (1967).

G.K. Engelis. *On some two–dimensional analogues of the classical orthogonal polynomials* (in Russian). Latviĭskiĭ Matematičeskiĭ Ežegodnik **15**, 169–202 (1974).

Multidimensional problems of approximation theory, numerical analysis, and probability theory require orthogonal polynomials of several discrete variables:

S. Karlin and J. McGregor. *Linear growth models with many types and multidimensional Hahn polynomials*. Theory Appl. spec. Funct., Proc. adv. Semin., Madison 1975, 261–288 (1975).

They formulate some genetic models in which the state space is multi–dimensional and discrete and they introduce the multivariable Hahn polynomials.

M.V. Tratnik. *Multivariable Meixner, Krawtchouk, and Meixner–Pollaczek polynomials*. J. Math. Phys. **30** (12), 2740–2749 (1989).

As already mentioned by Prof. Luc Vinet, Tratnik presented a multivariable biorthogonal generalization for Meixner, Kravchuk and Meixner–Pollaczek. Tratnik showed that these families of polynomials are orthogonal with respect to subspaces of lower degree and biorthogonal within a given subspace.

Moreover, in M.V. Tratnik. *Some multivariable orthogonal polynomials of the Askey tableau–discrete families.* J. Math. Phys. **32**(9), 2337–2342 (1991). an extension of the previously known multivariable Hahn polynomials to all of the remaining discrete families of the Askey squeme was given.
Do not forget that previously R.C. Griffiths, Orthogonal polynomials on the multinomial distribution, Austral. J. Statist. 13 (1971) 27-35. Corrigenda (1972) Austral. J. Statist. 14, 270.

Let us have a look to some of the papers of Tratnik.

Univariate (continuous) classical orthogonal polynomials can be characterized in terms of the second order linear differential equation of hypergeometric type

$$\sigma(x)y'' + \tau(x)y' + \lambda y = 0,$$

where $\sigma(x)$ and $\tau(x)$ are polynomials of at most the second and first degree respectively and λ is a constant. The solutions of the above equation have the property that derivatives of these solutions of any order also satisfy an equation of the same type.

A.F. Nikiforov, S.K. Suslov, and V.B. Uvarov. *Classical Orthogonal Polynomials of a Discrete Variable*. Springer Series in Computational Physics. (Springer, Berlin, 1991).

As a generalization, in Lyskova considered a special class of linear partial differential equations, called *basic class*,

$$\sum_{i,j=1}^{n} \tilde{a}_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} \tilde{b}_i(x) \frac{\partial u}{\partial x_i} + \lambda u = 0,$$

where $\tilde{a}_{ij}(x) = \tilde{a}_{ji}(x)$ and the coefficients $\tilde{a}_{ij}(x)$ and $\tilde{b}_i(x)$ are chosen so that the derivatives of any order of the solutions of the equation are also solutions of an equation of the same type.

A.S. Lyskova. *Orthogonal polynomials in several variables*. Sov. Math., Dokl. **43**(1), 264–268 (1991).

A.S. Lyskova. *On some properties of orthogonal polynomials in several variables*. Russ. Math. Surv. **52**(4), 840–841 (1997).

Let us have a look to the works of Lyskova

H.L. Krall and L.M. Sheffer. *Orthogonal polynomials in two variables*. Ann. Math. Pure Appl. **76** (1967), 325–376.

Krall and Sheffer studied the problem of finding all polynomial eigenfunctions of second order linear differential operators in two variables having polynomial coefficients of degree equal to the order of derivative under certain further restrictions relating to its symmetrizability and the orthogonality of their eigenfunctions. They classified all possible normal forms of the operators satisfying the required properties. It was shown by Vinet and Zhedanov that, for all these types, there correspond quantum mechanical systems on a Euclidean (pseudo–Euclidean) plane, two–dimensional sphere, or hyperboloid.

L. Vinet and A. Zhedanov. *Two–Dimensional Krall–Sheffer polynomials and quantum systems on spaces with constant curvature*. Letters in Mathematical Physics, **65**(2), 83–94 (2003).

Key concepts

Admissible: if there exists a sequence $\{\lambda_n\}$ ($n = 0, 1, \dots$) such that for $\lambda = \lambda_n$, there are precisely $n + 1$ linearly independent solutions in the form of polynomials of total degree n and has no non-trivial solutions in the set of polynomials whose total degree is less than n. This concept was introduced by Krall and Sheffer in the case of second order partial differential equations and also by Xu in the case of second order partial difference equations.

H.L. Krall, L.M. Sheffer, Orthogonal polynomials in two variables, Ann. Math. Pure Appl. 76 (1967) 325–376.

Y. Xu, Second order difference equations and discrete orthogonal polynomials of two variables, Internat. Math. Res. Notices 8 (2005) 449–475.

Framework

1. Admissible.
2. Hypergeometric.
3. Differential/difference/q-difference/divided-difference operators

From 1 to 2

One essential difference between polynomials in one variable and in several variables is the lack of an obvious basis in the latter.

One possibility to avoid this problem is to consider graded lexicographical order and use the matrix vector representation, first introduced by M.A. Kowalski (1982) and afterwards considered by Y. Xu (1993):

$$\mathbf{x}^n = (x^{n-k}y^k)^{\mathsf{T}}, \quad 0 \le k \le n, \quad n \in \mathbf{N}_0.$$

$$\mathbf{x}^1 = (x, y)^{\mathsf{T}}, \quad \mathbf{x}^2 = (x^2, xy, y^2)^{\mathsf{T}}, \ldots.$$

Let $\{P^n_{n-k,k}(x, y)\}$ be a sequence of polynomials in the space Π^2_n of all polynomials of total degree at most n in two variables, $\mathbf{x} = (x, y)$, with real coefficients. Such polynomials are finite sums of terms of the form $ax^{n-k}y^k$, where $a \in \mathbf{R}$.

From 1 to 2

From now on the (column) vector representation (Kowalski) will be adopted, so that $\mathbb{P}_n$ will denote the (column) polynomial vector

$$\mathbb{P}_n = (P^n_{n,0}(x,y), P^n_{n-1,1}(x,y), \ldots, P^n_{1,n-1}(x,y), P^n_{0,n}(x,y))^{\mathsf{T}}.$$

Then, each polynomial vector $\mathbb{P}_n$ can be written in terms of the $\mathbf{x}^n$ as:

$$\mathbb{P}_n = G_{n,n}\mathbf{x}^n + G_{n,n-1}\mathbf{x}^{n-1} + \cdots + G_{n,0}\,\mathbf{x}^0,$$

where $G_{n,j}$ are matrices of size $(n+1)\times(j+1)$ and the leading matrix coefficient $G_{n,n}$ is a nonsingular square matrix of size $(n+1)\times(n+1)$.

From 1 to 2

Definition (Monic polynomial vector)

A polynomial vector $\widehat{\mathbb{P}}_n$ is said to be monic if its leading matrix coefficient $\widehat{G}_{n,n}$ is the identity matrix (of size $(n+1)\times(n+1)$); i.e.:

$$\widehat{\mathbb{P}}_n = \mathbf{x}^n + \widehat{G}_{n,n-1}\mathbf{x}^{n-1} + \cdots + \widehat{G}_{n,0}\,\mathbf{x}^0 \,.$$

Then, each of its polynomial entries $\widehat{P}^n_{n-k,k}(x,y)$ are of the form:

$$\widehat{P}^n_{n-k,k}(x,y) = x^{n-k}y^k + \text{terms of lower total degree}\,.$$

“Hat" notation $\widehat{\mathbb{P}}_n$ will be used for monic polynomials.

For instance

$$\widehat{\mathbb{P}}_1 = \{x+a, y+b\}^{\mathsf{T}},$$

$$\widehat{\mathbb{P}}_2 = \{x^2+cx+dy+e, xy+fx+gy+h, y^2+mx+ny+u\}^{\mathsf{T}}$$

Bivariate orthogonality

Definition (Orthogonality)

Let $\mathcal{L}$ be a moment linear functional acting on Π_n^2. A sequence of polynomials $\{P_{n-k,k}^n(x,y)\} \subset \Pi_n^2$, is said to be orthogonal with respect to $\mathcal{L}$ if $\forall n \in \mathbb{N}_0$ there exist an invertible matrix H_n of size $n+1$ s.t.

$$\mathcal{L}\left[(\mathbf{x}^m \mathbb{P}_n^{\mathsf{T}})\right] = 0 \in \mathcal{M}^{(m+1,n+1)},\ n > m,\ \mathcal{L}\left[(\mathbf{x}^n \mathbb{P}_n^{\mathsf{T}})\right] = H_n \in \mathcal{M}^{(n+1,n+1)}$$

If there exists an integral representation of this orthogonality functional $\mathcal{L}$, then its action can be written in terms of a weight function $\varrho := \varrho(x,y)$ defined in a certain domain $D \subset \mathbb{R}^2$:

$$\mathcal{L}(P) = \iint_D P(x,y)\varrho(x,y)\,dx\,dy\,, \quad P \in \Pi_n^2\,,$$

which is defined in the set Π_n^2 provided that all the above integrals exist. Then, the family $\{\mathbb{P}_n\}_{n\geq 0}$ is said to be orthogonal with respect to ϱ in the domain D.

Some partial differential operators

Following Krall & Sheffer (1967) and Kwon, Lee & Littlejohn (2001) these are essentially the classical OP's in two variables:

1. $H_{n-k}(x)H_k(y)$, w.r.t. $\exp(-x^2-y^2)$ in $\mathbf{R}^2$, and
   $\mathcal{L} = \partial_{xx} + \partial_{yy} - 2(x\partial_x + y\partial_y)$.

2. $L^{\alpha}_{n-k}(x)L^{\beta}_{k}(y)$, w.r.t. $x^{\alpha}y^{\beta}e^{-x-y}$ in $(0,+\infty)^2$, and
   $\mathcal{L} = x\partial_{xx} + y\partial_{yy} + (1+\alpha-x)\partial_x + (1+\beta-y)\partial_y$.

3. $H_{n-k}(x)L^{\beta}_{k}(y)$, w.r.t. $y^{\beta}e^{-x^2-y}$ in $\mathbf{R}\times(0,+\infty)$, and
   $\mathcal{L} = \frac{1}{2}\partial_{xx} + y\partial_{yy} - x\partial_x + (1+\beta-y)\partial_y$.

1.

2.

3.

4. $P_{n-k}^{(\alpha,\beta+\gamma+2k+1)}(1-2x)(1-x)^k P_k^{(\beta,\gamma)}(1-2y/(1-x))$, w.r.t. $x^\alpha y^\beta (1-x-y)^\gamma$ in the triangular region $x, y > 0$, $x + y < 1$,
$\mathcal{L} = x(1-x)\partial_{xx} + y(1-y)\partial_{yy} - 2xy\partial_{xy}$
$+(\alpha+1-\varpi x)\partial_x + (\beta+1-\varpi y)\partial_y$, $\varpi = \alpha+\beta+\gamma+3$.

5. $P_{n-k}^{(\alpha+k+1/2,\alpha+k+1/2)}(x)(1-x^2)^{k/2} P_k^{(\alpha,\alpha)}(y(1-x^2)^{-1/2})$, w.r.t. $(1-x^2-y^2)^\alpha$ in the disk $x^2+y^2 < 1$, and
$\mathcal{L} = (1-x^2)\partial_{xx} + (1-y^2)\partial_{yy} - 2xy\partial_{xy} - (2\alpha+3)(x\partial_x + y\partial_y)$.

Let us have a look to a very classical book and highly recommended: P.K. Suetin, first edited in Russian in 1988 and translated into English in 1999

Starting from moments, some examples are given and later the so-called classical Appell's orthogonal polynomials are studied. Later, admissible differential equations for polynomials orthogonal over a domain is studied in detail.

Partial differential, difference and q-difference equations

Many families of orthogonal polynomials of a discrete variable (q-analogues and in nonuniform lattices were existing in the literature) so ...

$$\tilde{a}_{11}(x,y)\frac{\partial^2 u(x,y)}{\partial^2 x} + \tilde{a}_{12}(x,y)\frac{\partial^2 u(x,y)}{\partial x \partial y} + \tilde{a}_{22}(x,y)\frac{\partial^2 u(x,y)}{\partial^2 y} + \tilde{b}_1(x,y)\frac{\partial u(x,y)}{\partial x} + \tilde{b}_2(x,y)\frac{\partial u(x,y)}{\partial y} + \lambda u(x,y) = 0.$$

Using finite difference schemes we may approximate the first– and second–order partial derivatives, using the linear combination of the backward and forward difference quotients, with error $O(h^2) + O(k^2)$ for $h \to 0$ and $k \to 0$. The quantities $h > 0$ and $k > 0$ are the discretization parameters or mesh widths.

$$\frac{\partial u}{\partial x}(x,y) \approx \frac{1}{2}\left[\frac{u(x+h,y)-u(x,y)}{h}+\frac{u(x,y)-u(x-h,y)}{h}\right],$$
$$\frac{\partial u}{\partial y}(x,y) \approx \frac{1}{2}\left[\frac{u(x,y+k)-u(x,y)}{k}+\frac{u(x,y)-u(x,y-k)}{k}\right],$$

$$\frac{\partial^2 u}{\partial x^2}(x,y) \approx \frac{1}{h}\left[\frac{u(x+h,y)-u(x,y)}{h}-\frac{u(x,y)-u(x-h,y)}{h}\right],$$
$$\frac{\partial^2 u}{\partial y^2}(x,y) \approx \frac{1}{k}\left[\frac{u(x,y+k)-u(x,y)}{k}-\frac{u(x,y)-u(x,y-k)}{k}\right],$$

$$\frac{\partial^2 u}{\partial x \partial y}(x,y) \approx \frac{1}{2k}\left[\frac{u(x+h,y)-u(x,y)}{h}-\frac{u(x+h,y-k)-u(x,y-k)}{h}\right]$$
$$+\frac{1}{2h}\left[\frac{u(x,y+k)-u(x,y)}{k}-\frac{u(x-h,y+k)-u(x-h,y)}{k}\right]$$

For $h = k = 1$ we obtain

$$\begin{aligned}\sigma_{11}(x,y)\Delta_1\nabla_1 u(x,y) + \sigma_{12}(x,y)\Delta_1\nabla_2 u(x,y) \\ + \sigma_{21}(x,y)\Delta_2\nabla_1 u(x,y) + \sigma_{22}(x,y)\Delta_2\nabla_2 u(x,y) \\ + \tau_1(x,y)\Delta_1 u(x,y) + \tau_2(x,y)\Delta_2 u(x,y) + \lambda u(x,y) = 0\,.\end{aligned}$$

Partial differential, difference and q-difference equations

$$\tilde{a}_{11}(x,y)\frac{\partial^2 u(x,y)}{\partial^2 x} + \tilde{a}_{12}(x,y)\frac{\partial^2 u(x,y)}{\partial x \partial y} + \tilde{a}_{22}(x,y)\frac{\partial^2 u(x,y)}{\partial^2 y} + \tilde{b}_1(x,y)\frac{\partial u(x,y)}{\partial x} + \tilde{b}_2(x,y)\frac{\partial u(x,y)}{\partial y} + \lambda u(x,y) = 0.$$

This equation uses the simplest q-difference schemes of the second-order precision

$$a_{11}(x,y)D_q^1 D_{q^{-1}}^1 u(x,y) + a_{22}(x,y)D_q^2 D_{q^{-1}}^2 u(x,y) + a_{12a}(x,y)D_q^1 D_q^2 u(x,y) + a_{12d}(x,y)D_{q^{-1}}^1 D_{q^{-1}}^2 u(x,y) + b_1(x,y)D_q^1 u(x,y) + b_2(x,y)D_q^2 u(x,y) + \lambda u(x,y) = 0.$$

Hypergeometric equations

Definition

We say that the partial differential equation is hypergeometric if all the derivatives $u_\alpha(\mathbf{x}) = D_1^r D_2^s u(\mathbf{x})$ of the solutions $u = u(\mathbf{x})$ of the equation are also solutions of an equation of the same type.

This concept was introduced by Lyskova (1991) as basic class in the multivariable continuous case. In the discrete, $u_\alpha(\mathbf{x}) = \Delta_1^r \Delta_2^s u(\mathbf{x})$. In the q-case, $u_\alpha(\mathbf{x}) = [D_q^1]^r [D_q^2]^s u(\mathbf{x})$.

This concept applied to the equation implies some restrictions on the coefficients a_{ij} and b_i. For instance, in the continuous case

$$a_{11}(x,y) = a_{11}(x) = a_1 x^2 + b_1 x + c_1 + 0y^2 + 0xy + 0y,$$
$$b_1(x,y) = b_1(x) = f_1 x + g_1 + 0y$$
$$a_{12}(x,y) = 0x^2 + 0y^2 + s_{12}xy + t_{12}x + u_{12}y + v_{12}$$

Hypergeometric and admissible equation

Definition

The hypergeometric difference equation will be called admissible if there exists a sequence $\{\lambda_n\}$ $(n = 0, 1, \dots)$ such that for $\lambda = \lambda_n$, there are precisely $n + 1$ linearly independent solutions in the form of polynomials of total degree n and has no non-trivial solutions in the set of polynomials whose total degree is less than n.

This condition imposed on the equation implies some restrictions on the coefficients of the polynomials a_{ij} and b_i. For instance, in the continuous case we have

$$a_{11}(x) = ax^2 + b_1 x + c_1, \quad a_{22}(y) = ay^2 + b_2 y + c_2$$
$$a_{12}(x, y) = axy + \cdots, \quad b_1(x) = fx + g_1, \quad b_2(y) = fy + g_2$$

Potentially self-adjoint equation

$$
\begin{aligned}
\mathcal{D}u(\mathbf{x}) = \sigma_{11}(\mathbf{x})\Delta_1\nabla_1 u(\mathbf{x}) \\
&+ \sigma_{12}(\mathbf{x})\Delta_1\nabla_2 u(\mathbf{x}) + \sigma_{21}(\mathbf{x})\Delta_2\nabla_1 u(\mathbf{x}) \\
&+ \sigma_{22}(\mathbf{x})\Delta_2\nabla_2 u(\mathbf{x}) + \tau_1(\mathbf{x})\Delta_1 u(\mathbf{x}) + \tau_2(\mathbf{x})\Delta_2 u(\mathbf{x})
\end{aligned}
$$

The adjoint operator $\mathcal{D}^\dagger$ of $\mathcal{D}$ is defined by

$$
\begin{aligned}
\mathcal{D}^\dagger u = \Delta_1\nabla_1(\sigma_{11}u) + \Delta_1\nabla_2(\sigma_{21}u) + \Delta_2\nabla_1(\sigma_{12}u) + \Delta_2\nabla_2(\sigma_{22}u) \\
- \nabla_1(\tau_1 u) - \nabla_2(\tau_2 u).
\end{aligned}
$$

The operator $\mathcal{D}$ is self-adjoint if $\mathcal{D}^\dagger = \mathcal{D}$.

Potentially self-adjoint operators

Definition

The operator $\mathcal{D}$ is potentially self-adjoint in a domain G if there exists in this domain a positive real function $\varrho(\mathbf{x}) = \varrho(x, y)$ such that the operator $\varrho(\mathbf{x})\mathcal{D}$ is self-adjoint in the domain G.

In order that $\mathcal{D}$ be potentially self-adjoint, we multiply the equation through by a positive function $\varrho(\mathbf{x})$ in some domain G. Then, the operator is self-adjoint provided [RGA, 2007] the following conditions

Potentially self-adjoint operators

$$
\begin{aligned}
\frac{\varrho(x,y)}{\varrho(x+1,y)} &= \frac{\sigma_{11}(x+1,y)+\sigma_{21}(x+1,y)}{\sigma_{11}(x,y)+\sigma_{12}(x,y)+\tau_1(x,y)},\\
\frac{\varrho(x,y)}{\varrho(x+1,y-1)} &= \frac{\sigma_{21}(x+1,y-1)}{\sigma_{12}(x,y)},\\
\frac{\varrho(x,y)}{\varrho(x,y+1)} &= \frac{\sigma_{12}(x,y+1)+\sigma_{22}(x,y+1)}{\sigma_{21}(x,y)+\sigma_{22}(x,y)+\tau_2(x,y)}.
\end{aligned}
$$

$$
\begin{aligned}
\Delta_1\left[\sigma_{11}\varrho\right]+\Delta_2\left[\sigma_{12}\varrho\right] &= \tau_1\varrho,\\
\Delta_1\left[\sigma_{21}\varrho\right]+\Delta_2\left[\sigma_{22}\varrho\right] &= \tau_2\varrho,\\
\Delta_1\nabla_2\left[\sigma_{21}\varrho\right] &= \Delta_2\nabla_1\left[\sigma_{12}\varrho\right].
\end{aligned}
$$

Weight function for orthogonality

$$
\begin{aligned}
\varpi_1(x,y) &= \sigma_{11}(x,y) + \sigma_{21}(x,y),\\
\varpi_2(x,y) &= \sigma_{22}(x,y) + \sigma_{12}(x,y),\\
\varpi_3(x,y) &= \sigma_{11}(x,y) + \sigma_{12}(x,y) + \tau_1(x,y),\\
\varpi_4(x,y) &= \sigma_{21}(x,y) + \sigma_{22}(x,y) + \tau_2(x,y).
\end{aligned}
$$

Then,

$$
\mathcal{G}_1(x,y) = \frac{\varpi_3(x,y)}{\varpi_1(x+1,y)}, \quad \mathcal{G}_2(x,y) = \frac{\varpi_4(x,y)}{\varpi_2(x,y+1)},
$$

$$
\varrho(x,y) = \kappa \prod_{i=y_0}^{y-1} \mathcal{G}_2(x,i) \prod_{j=x_0}^{x-1} \mathcal{G}_1(j,y_0)\,.
$$

Example

The linear partial difference equation

$$
\begin{aligned}
(p_1-1)x\Delta_1\nabla_1 u(x,y) + (p_2-1)y\Delta_2\nabla_2 u(x,y) \\
+ p_1 y\Delta_1\nabla_2 u(x,y) + p_2 x\Delta_2\nabla_1 u(x,y) \\
+ (x-Np_1)\Delta_1 u(x,y) + (y-Np_2)\Delta_2 u(x,y) - (n_1+n_2)u(x,y) = 0\,,
\end{aligned}
$$

has as a solution the bivariate Kravchuk polynomials of total degree n_1+n_2, defined by Tratnik as the generalized Kampé de Fériet hypergeometric series

$$
\begin{aligned}
K_{n_1,n_2}^{p_1,p_2}(x,y;N) = (x+y-N)_{n_1+n_2} \\
\times F_{1:0;0}^{0:2;2}\left(\begin{matrix} - : -n_1, -x; -n_2, -y \\ -n_1-n_2-x-y+N+1 : -; - \end{matrix} \;\middle|\; \frac{p_1+p_2-1}{p_1}, \frac{p_1+p_2-1}{p_2} \right)
\end{aligned}
$$

where N is a non–negative integer and p_1, p_2 are real parameters satisfying $p_1>0, \quad p_2>0, \quad 0<p_1+p_2<1.$

$$\varrho(x,y) = \kappa \prod_{i=y_0}^{y-1} \mathcal{G}_2(x,i) \prod_{j=x_0}^{x-1} \mathcal{G}_1(j,y_0)\,.$$

For this specific equation, the approach gives us as orthogonality weight function

$$\varrho(x,y) = \frac{N!}{x!y!(N-x-y)!} p_1^x p_2^y (1-p_1-p_2)^{N-x-y}$$

which is the trinomial distribution (bivariate extension of binomial distribution), in the triangular domain G given by

$$x \geq 0, \quad y \geq 0, \quad 0 \leq x+y \leq N,$$

with $p_1 > 0, \quad p_2 > 0, \quad 0 < p_1 + p_2 < 1$.

The monic bivariate Kravchuk polynomials (Tratnik 1991)

$$\hat{K}_{n_1,n_2}^{p_1,p_2}(x,y;N) = (-1)^{n_1+n_2} p_1^{n_1} p_2^{n_2} (N-n_1-n_2+1)_{n_1+n_2} \times F_{1:0;0}^{0:2;2}\left(\begin{array}{c} -:-n_1,-x;-n_2,-y \\ -N:-;- \end{array} \middle| \frac{1}{p_1}, \frac{1}{p_2} \right),$$

the non-monic bivariate Kravchuk polynomials (Tratnik 1991), defined as a generalized Kampé de Fériet series

$$K_{n_1,n_2}^{p_1,p_2}(x,y;N) = (x+y-N)_{n_1+n_2} \times F_{1:0;0}^{0:2;2}\left(\begin{array}{c} -:-n_1,-x;-n_2,-y \\ -n_1-n_2-x-y+N+1:-;- \end{array} \middle| \frac{p_1+p_2-1}{p_1}, \frac{p_1+p_2-1}{p_2} \right)$$

the non-monic bivariate Kravchuk polynomials defined as a product of Kravchuk polynomials

$$\tilde{K}^{p_1,p_2}_{n_1,n_2}(x,y;N) = \frac{(N-n_1)!}{N!\,(n_1-N)_{n_2}} K_{n_1}(x;p_1/(p_1+p_2),x+y)\,K_{n_2}(x+y-n_1;p_1+p_2,N-n_1)\,,$$

where for $0<p<1$ and $n=0,1,\ldots,N$, the univariate Kravchuk polynomials are normalized as

$$K_n(x;p,N) = (-N)_n\,{}_2F_1\left(\begin{matrix}-n,-x\\-N\end{matrix}\,\middle|\,\frac{1}{p}\right),$$

and the non-monic bivariate Kravchuk polynomials (Geronimo and Iliev, 2010) defined also as a product of univariate Kravchuk polynomials

$$K_2(n_1,n_2;x,y;p_1,p_2;N) = \frac{1}{(-N)_{n_1+n_2}} K_{n_1}(x;p_1,N-n_2)\,K_{n_2}(y;p_2/(1-p_1),N-x).$$

Also ... as Rodrigues formula!

Partial differential equations

So, in the continuous, discrete and even for the *q*-analogues we are considering equations of this type

$$\tilde{a}_{11}(x,y)\frac{\partial^2 u(x,y)}{\partial^2 x} + \tilde{a}_{12}(x,y)\frac{\partial^2 u(x,y)}{\partial x \partial y} + \tilde{a}_{22}(x,y)\frac{\partial^2 u(x,y)}{\partial^2 y} + \tilde{b}_1(x,y)\frac{\partial u(x,y)}{\partial x} + \tilde{b}_2(x,y)\frac{\partial u(x,y)}{\partial y} + \lambda_n u(x,y) = 0,$$

which we assume that are hypergeometric, admissible and potentially self-adjoint.

Why to restrict to this class? There are families out of here! (e.g. in the book of Suetin). First answer from a quite recent paper. Second answer at the end of the talk.

Recurrence relations

The idea is to use the (column) vector representation (Kowalski).

$$\mathbb{P}_n = (P^n_{n,0}(x,y), P^n_{n-1,1}(x,y), \ldots, P^n_{1,n-1}(x,y), P^n_{0,n}(x,y))^{\mathsf{T}}.$$

$$\mathbb{P}_n = G_{n,n}\mathbf{x}^n + G_{n,n-1}\mathbf{x}^{n-1} + \cdots + G_{n,0}\,\mathbf{x}^0,$$

where $G_{n,j}$ are matrices of size $(n+1)\times(j+1)$ and $G_{n,n}$ is a nonsingular square matrix of size $(n+1)\times(n+1)$.
As mentioned some slides before, monic if its leading matrix coefficient $\widehat{G}_{n,n}$ is the identity matrix (of size $(n+1)\times(n+1)$); i.e.:

$$\widehat{\mathbb{P}}_n = \mathbf{x}^n + \widehat{G}_{n,n-1}\mathbf{x}^{n-1} + \cdots + \widehat{G}_{n,0}\,\mathbf{x}^0\,.$$

Then, each of its polynomial entries $\widehat{P}^{n}_{n-k,k}(x,y)$ are of the form:

$$\widehat{P}^{n}_{n-k,k}(x,y) = x^{n-k}y^{k} + \text{terms of lower total degree}\,.$$

Known results

Theorem (Dunkl and Xu)

Let $\mathcal{L}$ be a positive definite moment linear functional acting on the space Π_n^2 of all polynomials of total degree at most n in two variables, and $\{\mathbb{P}_n\}_{n\geq 0}$ be an orthogonal family with respect to $\mathcal{L}$. Then, for $n \geq 0$, there exist unique matrices $A_{n,j}$ of size $(n+1)\times(n+2)$, $B_{n,j}$ of size $(n+1)\times(n+1)$,and $C_{n,j}$ of size $(n+1)\times n$, such that

$$x_j\mathbb{P}_n = A_{n,j}\mathbb{P}_{n+1} + B_{n,j}\mathbb{P}_n + C_{n,j}\mathbb{P}_{n-1}, \quad j = 1,2$$

with the initial conditions $\mathbb{P}_{-1} = 0$ and $\mathbb{P}_0 = 1$. Here the notation $x_1 = x$, $x_2 = y$ is used.

Theorem

The explicit expressions of the matrices $A_{n,j}$, $B_{n,j}$ and $C_{n,j}$ ($j = 1, 2$) in terms of the values of the leading coefficients $G_{n,n}$, $G_{n,n-1}$ and $G_{n,n-2}$ in the expansions are given by

$$\begin{cases} A_{n,j} = G_{n,n} L_{n,j} G^{-1}_{n+1,n+1}, & n \geq 0, \\ B_{0,j} = -A_{0,j} G_{1,0}, \\ B_{n,j} = (G_{n,n-1} L_{n-1,j} - A_{n,j} G_{n+1,n}) G^{-1}_{n,n}, & n \geq 1, \\ C_{1,j} = -(A_{1,j} G_{2,0} + B_{1,j} G_{1,0}), \\ C_{n,j} = (G_{n,n-2} L_{n-2,j} - A_{n,j} G_{n+1,n-1} - B_{n,j} G_{n,n-1}) G^{-1}_{n-1,n-1}, & n \geq 2\,. \end{cases}$$

The matrices $L_{n,j}$ of the size $(n+1)\times(n+2)$

$$L_{n,1} = \begin{pmatrix} 1 & & \bigcirc & 0 \\ & \ddots & & \vdots \\ \bigcirc & & 1 & 0 \end{pmatrix} \quad \text{and} \quad L_{n,2} = \begin{pmatrix} 0 & 1 & & \bigcirc \\ \vdots & & \ddots & \\ 0 & \bigcirc & & 1 \end{pmatrix},$$

so that

$$x\,\mathbf{x}^n = L_{n,1}\mathbf{x}^{n+1}, \quad y\,\mathbf{x}^n = L_{n,2}\mathbf{x}^{n+1},$$

and

$$x^2\,\mathbf{x}^n = L_{n,1}L_{n+1,1}\mathbf{x}^{n+2}, \quad y^2\,\mathbf{x}^n = L_{n,2}L_{n+1,2}\mathbf{x}^{n+2},$$
$$L_{n,2}L_{n+1,1} = L_{n,1}L_{n+1,2},$$

and for $j=1,2$,

$$L_{n,j}\,L_{n,j}^{\mathsf{T}} = I_{n+1},$$

where I_{n+1} denotes the identity matrix of the size $n+1$.

Moreover,

$$\partial_j \mathbf{x}^n = \mathbb{E}_{n,j}\, \mathbf{x}^{n-1}, \qquad j = 1, 2,$$

where the matrices $\mathbb{E}_{n,j}$ of the size $(n+1) \times n$ are given by

$$\mathbb{E}_{n,1} = \begin{pmatrix} n & & & \bigcirc \\ & n-1 & & \\ & & \ddots & \\ & \bigcirc & & 1 \\ 0 & \dots & 0 & 0 \end{pmatrix}, \qquad \mathbb{E}_{n,2} = \begin{pmatrix} 0 & \dots & & 0 \\ 1 & & & \bigcirc \\ & 2 & & \\ & & \ddots & \\ & \bigcirc & & n \end{pmatrix}.$$

Discrete situation

Explicit expression for the matrices $\widehat{G}_{n,n-1}$ and $\widehat{G}_{n,n-2}$:

$$\widehat{G}_{n,n-1} = \mathbb{S}_n \mathbb{F}^{-1}_{n-1}(\lambda_n),$$
$$\widehat{G}_{n,n-2} = \left(\mathbb{T}_n + \widehat{G}_{n,n-1}\mathbb{S}_{n-1}\right) \mathbb{F}^{-1}_{n-2}(\lambda_n),$$

where the nonsingular matrix $\mathbb{F}_n(\lambda_\ell)$ is given by

$$\mathbb{F}_n(\lambda_\ell) = (\lambda_n - \lambda_\ell)\mathbb{I}_{n+1},$$

$\mathbb{I}_{n+1}$ denotes the identity matrix of size $(n+1) \times (n+1)$

Moreover, using the left inverse $D_n^{\dagger}$ of the joint matrix L_n

$$D_n^{\dagger} = \begin{pmatrix} 1 & & & & 0 & & & \\ & 1/2 & & \bigcirc & 1/2 & & \bigcirc & \\ & & \ddots & & & \ddots & & \\ & \bigcirc & & 1/2 & \bigcirc & & 1/2 & \\ & & & & 0 & & & 1 \end{pmatrix},$$

we can write a recursive formula for the monic orthogonal polynomials

$$\widehat{\mathbb{P}}_{n+1} = D_n^{\dagger}\left[\begin{pmatrix} x \\ y \end{pmatrix} \otimes I_{n+1} - B_n\right]\widehat{\mathbb{P}}_n - D_n^{\dagger} C_n \widehat{\mathbb{P}}_{n-1}, \quad n \geq 0,$$

with the initial conditions $\widehat{\mathbb{P}}_{-1} = 0$, $\widehat{\mathbb{P}}_0 = 1$, where $\otimes$ denotes the Kronecker product and

$$B_n = \left(B_{n,1}^{\mathsf{T}}, B_{n,2}^{\mathsf{T}}\right)^{\mathsf{T}}, \quad C_n = \left(C_{n,1}^{\mathsf{T}}, C_{n,2}^{\mathsf{T}}\right)^{\mathsf{T}},$$

are matrices of size $(2n+2) \times (n+1)$ and $(2n+2) \times n$, respectively.

Bivariate big q-Jacobi polynomials

Non-monic

$$P_{n,k}(x,y;a,b,c,d;q) := P_{n-k}(y;a,bcq^{2k+1},dq^k;q) \times y^k (dq/y;q)_k P_k(x/y;c,b,d/y;q),$$

$$n \in \mathbb{N}, \quad k = 0,1,\dots,n, \quad q \in (0,1), \quad 0 < aq, bq, cq < 1, \quad d < 0,$$

where the univariate big q-Jacobi are (under some restrictions on the parameters)

$$P_m(t;A,B,C;q) := {}_3\phi_2\left(\begin{matrix} q^{-m}, ABq^{m+1}, t \\ Aq, Cq \end{matrix} \,\middle|\, q;q \right).$$

Potentially self-adjoint admissible second order partial q-difference equation of hypergeometric type

$$\begin{cases} a_{11}(x) = \sqrt{q}\,(d\,q - x)\,\left(a\,c\,q^2 - x\right), a_{22}(y) = \sqrt{q}\,(a\,q - y)\,(d\,q - y), \\ a_{12a}(x,y) = a\,c\,q^4\,(d - b\,x)\,(1 - y), a_{12d}(x,y) = (d\,q - x)\,(a\,q - y), \\ [...]. \end{cases}$$

Bivariate big q-Jacobi polynomials

Lewanowicz and Wozny proved that these polynomials satisfy the following orthogonality relation

$$\int_{dq}^{aq}\int_{dq}^{cqy} W(x,y;a,b,c,d;q)P_{n,k}(x,y;q)P_{m,l}(x,y;q)d_qxd_qy \\ = H_{n,k}(a,b,c,d;q)\delta_{n,m}\delta_{k,l},$$

where $0 < aq, bq, cq < 1$, $d < 0$, and the weight function is defined by

$$W(x,y;a,b,c,d;q) := \frac{(dq/y, c^{-1}x/y, x/d, y/a, y/d;q)_\infty}{y(c^{-1}d/y, cqy/d, x/y, bx/d, y;q)_\infty},$$

with the notations

$$(a_1,\ldots,a_r;\, q)_\infty = (a_1;\, q)_\infty \cdots (a_r;\, q)_\infty,$$

and

$$(a;\, q)_\infty = \prod_{j=0}^{\infty}(1-aq^j).$$

Monic solutions

$B_{n,1}$

$$\begin{cases} \dfrac{dq^{-i+n+2}\left(acq^{2i-1}\left(q^{-i+n+1}\left(b\left(acq^{i+n+1}+q^{-i+n+2}-q-1\right)-q-1\right)+1\right)+1\right)}{\left(abcq^{2n+1}-1\right)\left(abcq^{2n+3}-1\right)} \\ +\dfrac{acq^{n+2}\left(-b(q+1)q^{n-i}\left(acq^{2i}+q\right)+ab(b+1)cq^{2n+2}+b+1\right)}{\left(abcq^{2n+1}-1\right)\left(abcq^{2n+3}-1\right)}, \quad (i=j), \\ -\dfrac{\left(q^{i-1}-1\right)\left(aq^{i-1}-1\right)q^{-i+n+2}\left(abcdq^{2n+2}-abc(q+1)q^{n+1}+d\right)}{\left(abcq^{2n+1}-1\right)\left(abcq^{2n+3}-1\right)}, \quad (i=j+1), \\ 0, \quad \text{otherwise}, \end{cases}$$

$B_{n,2}$

$$\begin{cases} q^{-i}\left(\dfrac{q[i]_q\left(aq^{i}-1\right)\left(dq^{n+1}-1\right)}{abcq^{2n+3}-1}+\dfrac{q[i-1]_q\left(q-aq^{i}\right)\left(dq^{n}-1\right)}{abcq^{2n+1}-1}+q\right), \quad (i=j), \\ -\dfrac{acq^{i+1}\left(q^{-i+n+1}-1\right)\left(bq^{-i+n+1}-1\right)\left(abcq^{2(n+1)}-d(q+1)q^{n}+1\right)}{\left(abcq^{2n+1}-1\right)\left(abcq^{2n+3}-1\right)}, \quad (i=j-1), \\ 0, \quad \text{otherwise}, \end{cases}$$

Monic solutions

$$\widehat{P}_{n,m}(x,y;a,b,c,d;q) = \frac{\left(\frac{d}{b}\right)^n (aq;\,q)_m(bq;\,q)_n(dq^{n+1};\,q)_m(abcq^{m+2}/d;\,q)_n}{(abcq^{m+n+2};\,q)_{n+m}}$$

$$\times \sum_{i=0}^{n}\sum_{j=0}^{m} \frac{(-1)^{-i-j}\begin{bmatrix} n \\ i \end{bmatrix}_q \begin{bmatrix} m \\ j \end{bmatrix}_q q^{\frac{1}{2}(i(i-2n+1)+j(j-2m+1))}(abcq^{m+n+2};\,q)_{i+j}}{(aq;\,q)_j(bq;\,q)_i(dq^{n+1};\,q)_j(abcq^{m+2}/d;\,q)_i}(bx/d;\,q)_i(y;\,q)_j$$

$$= \frac{\left(\frac{d}{b}\right)^n (aq;\,q)_m(bq;\,q)_n(dq^{n+1};\,q)_m(abcq^{m+2}/d;\,q)_n}{(abcq^{m+n+2};\,q)_{n+m}}$$

$$\times\ \Phi^{1:2;2}_{0:2;2}\left[\begin{array}{c} abcq^{n+m+2} : q^{-n}, bx/d; q^{-m}, y \\ - : bq, abcq^{m+2}/d; aq, dq^{n+1} \end{array} \middle| \begin{array}{c} q : q, q \\ 0, 0, 0 \end{array}\right],$$

where the generalized bivariate basic hypergeometric series is defined by

$$\Phi^{\lambda:r;s}_{\mu:u;v}\left[\begin{array}{c} \alpha_1,\ldots,\alpha_\lambda : a_1,\ldots,a_r; c_1,\ldots,c_s \\ \beta_1,\ldots,\beta_\mu : b_1,\ldots,b_u; d_1,\ldots,d_v \end{array} \middle| \begin{array}{c} q : x, y \\ i, j, k \end{array}\right]$$

$$= \sum_{m,n=0}^{\infty} \frac{(\alpha_1,\ldots,\alpha_\lambda;\,q)_{m+n}(a_1,\ldots,a_r;\,q)_m(c_1,\ldots,c_s;\,q)_n}{(\beta_1,\ldots,\beta_\mu;\,q)_{m+n}(b_1,\ldots,b_u;\,q)_m(d_1,\ldots,d_v;\,q)_n}\,\frac{x^m y^n q^{i\binom{m}{2}+j\binom{n}{2}+kmn}}{(q;\,q)_m\,(q;\,q)_n}.$$

Third solution

$$\tilde{P}_{n,m}(x,y;a,b,c,d;q) = \frac{\Lambda_{n,m}}{\varrho(x,y)}[D_q^1]^{(n)}[D_q^2]^{(m)}$$
$$\left[\varrho(x,y)x^{2n}y^{2m}(dq/x;\,q)_n(aq/y;\,q)_m(x/y;\,q)_m(cqy/x;\,q)_n\right],$$

Limit as $q \uparrow 1$

$$\left(x^2-1\right)\frac{\partial^2}{\partial x^2}f+\left(y^2-1\right)\frac{\partial^2}{\partial y^2}f+2\left((x+1)(y-1)\right)\frac{\partial^2}{\partial x\partial y}f$$
$$+\left(x(\alpha+\beta+\gamma+3)+\alpha-\beta+\gamma+1\right)\frac{\partial}{\partial x}f$$
$$+\left(y(\alpha+\beta+\gamma+3)+\alpha-\beta-\gamma-1\right)\frac{\partial}{\partial y}f$$
$$-n(\alpha+\beta+\gamma+n+2)f=0.$$

The orthogonality weight function for the polynomial solutions of the above equation can be computed following AGRZ giving rise to

$$\varrho^{(\alpha,\beta,\gamma)}(x,y)=(1-y)^{\alpha}(x+1)^{\beta}(y-x)^{\gamma},$$

in the triangular domain

$$\mathcal{R}=\{(x,y)\in\mathbf{R}^2\,|\,x\le y\le 1,\ \ -1\le x\le 1\}.$$

At least three orthogonal polynomial solutions of the latter PDE with respect to the same weight function $\varrho^{(\alpha,\beta,\gamma)}(x,y)$ on the same domain $\mathcal{R}$

1. The monic polynomial solutions of the latter PDE satisfy three-term recurrence relations where the matrix coefficients can be easily computed by considering the limit as $q \uparrow 1$ of our matrices for $a = q^{\alpha}$, $b = q^{\beta}$, $c = q^{\gamma}$ and $d = -q^{\delta}$, or eventually from AGRZ.
   Moreover, we have the following representation in terms of generalized Kampé de Fériet hypergeometric series as

$$\widehat{\mathrm{A}}_{n,m}^{(\alpha,\beta,\gamma)}(x,y) = (-1)^n 2^{n+m} \frac{(\alpha+1)_m\,(\beta+1)_n}{(\alpha+\beta+\gamma+m+n+2)_{n+m}}$$
$$\times\ F^{1:1;1}_{0:1;1}\left(\begin{array}{c} \alpha+\beta+\gamma+m+n+2 : -n; -m \\ - : \beta+1; \alpha+1 \end{array} \middle| \frac{x+1}{2}, \frac{1-y}{2}\right).$$

Partial differential equation

1. Monic
2. The non-monic polynomials which can be computed from the Rodrigues formula (exactly coincide with those in the book of Suetin, Chapter III)

$$\varrho^{(\alpha,\beta,\gamma)}(x,y)\,\tilde{A}_{n,m}^{(\alpha,\beta,\gamma)}(x,y) = \frac{\partial^{n+m}}{\partial x^n\,\partial y^m}\left[(x+1)^{\beta+n}(1-y)^{\alpha+m}(y-x)^{\gamma+n+m}\right].$$

Partial differential equation

1. Monic (recurrence relation in matrix form and explicit rep.)
2. Non-monic Rodrigues's
3. And a third non-monic polynomial solution

$$J_{n,m}(x,y;\alpha,\beta,\gamma) = \frac{m!(y+1)^m(n-m)!}{(\gamma+1)_m(\alpha+1)_{n-m}} \times P_m^{(\gamma,\beta)}\left(\frac{2x-y+1}{y+1}\right) P_{n-m}^{(\alpha,\beta+\gamma+2m+1)}(y),$$

where for $a, b > -1$,

$$P_n^{(a,b)}(x) = \frac{(a+1)_n}{n!}\, {}_2F_1\left(\begin{matrix} -n, n+a+b+1 \\ a+1 \end{matrix} \;\middle|\; \frac{1-x}{2}\right),$$

are the (classical and univariate) Jacobi polynomials.

Limits!!!

1. $\lim_{q \uparrow 1} W(x, y; q^\alpha, q^\beta, q^\gamma, -q^\delta; q) = \varrho^{(\alpha,\beta,\gamma)}(x, y)$.
2. Matrices $\uparrow 1$ Matrices!!
3. Monic
$$\lim_{q \to 1} \widehat{P}_{n,m}(x, y; q^\alpha, q^\beta, q^\gamma, -q^\delta; q) = \widehat{\mathrm{A}}_{n,m}^{(\alpha,\beta,\gamma)}(x, y).$$
4. Non-monic
$$\lim_{q \to 1} P_{n,m}(x, y; q^\alpha, q^\beta, q^\gamma, -q^\delta; q) = J_{n,m}(x, y; \alpha, \beta, \gamma).$$
5. Non monic (Rodrigues')
$$\lim_{q \to 1} \tilde{P}_{n,m}(x, y; q^\alpha, q^\beta, q^\gamma, -q^\delta; q) = \tilde{A}_{n,m}^{(\alpha,\beta,\gamma)}(x, y).$$

In this framework we know that from the second-order linear partial differential equation of hypergeometric type we obtain

1. Three-term recurrence relation
2. Rodrigues formula
3. Pearson's system
4. Structure relation(s)
5. ...

Next step: nonuniform lattices.

Extreme difficulties appear.

The theory of difference analogues of special functions of hypergeometric type

S.K. Suslov

CONTENTS

In the following exposition of the Calculus of Finite Differences, particular attention has been paid to the connection of its methods with those of the Differential Calculus—a connection which in some instances involves far more than a merely formal analogy.

G. Boole

18 April 1860

(From the preface to the first edition of "A Treatise on the Calculus of Finite Differences".)

Nonuniform lattices

Univariate Racah polynomials can be defined in terms of hypergeometric series as

$$r_n(\alpha,\beta,\gamma,\delta;s) = r_n(s) = (\alpha+1)_n\,(\beta+\delta+1)_n\,(\gamma+1)_n$$
$$\times\ {}_4F_3\left(\begin{matrix}-n, n+\alpha+\beta+1, -s, s+\gamma+\delta+1\\ \alpha+1, \beta+\delta+1, \gamma+1\end{matrix}\,\middle|\,1\right), \quad n=0,1,\ldots,N,$$

where $r_n(\alpha,\beta,\gamma,\delta;s)$ is a polynomial of degree $2n$ in s and of degree n in the quadratic lattice

$$\eta(s) = s(s+\gamma+\delta+1),$$

and $(A)_n = A(A+1)\cdots(A+n-1)$ with $(A)_0 = 1$ denotes the Pochhammer symbol.

Univariate Racah polynomials satisfy the following second-order linear divided-difference equation

$$\phi(\eta(s))\mathbb{D}_\eta^2 r_n(s) + \tau(\eta(s))\mathbb{S}_\eta\mathbb{D}_\eta r_n(s) + \lambda_n r_n(s) = 0,$$

where ϕ is a polynomial of degree two in the lattice $\eta(s)$ given by

$$\begin{aligned}\phi(\eta(s)) &= -(\eta(s))^2 \\ &+ \frac{1}{2}(-\alpha(2\beta+\delta+\gamma+3)+\beta(\delta-\gamma-3)-2(\delta\gamma+\delta+\gamma+2))\eta(s) \\ &\qquad - \frac{1}{2}(\alpha+1)(\gamma+1)(\beta+\delta+1)(\delta+\gamma+1),\end{aligned}$$

τ is a polynomial of degree one in the lattice $\eta(s)$ given by

$$\tau(\eta(s)) = -(\alpha+\beta+2)\eta(s) - (\alpha+1)(\gamma+1)(\beta+\delta+1),$$

the eigenvalues λ_n are given by

$$\lambda_n = n(\alpha+\beta+n+1),$$

and the difference operators $\mathbb{D}_\eta$ and $\mathbb{S}_\eta$ are defined by

$$\mathbb{D}_\eta f(s) = \frac{f(s+1/2) - f(s-1/2)}{\eta(s+1/2) - \eta(s-1/2)}, \ \mathbb{S}_\eta f(s) = \frac{f(s+1/2) + f(s-1/2)}{2}.$$

Notice that the above operators transform polynomials of degree n in the lattice $\eta(s)$ into polynomials of respectively degree $n-1$ and n in the same variable $\eta(s)$.

We would like to notice here that

$$\begin{aligned}\mathbb{D}_\eta r_n(\alpha, \beta, \gamma, \delta; s) \\ = n(n+\alpha+\beta+1) r_{n-1}(\alpha+1, \beta+1, \gamma+1, \delta; s-1/2)\end{aligned}$$

Multivariable Racah polynomials have been introduced by Tratnik and deeply analyzed by Geronimo and Iliev, where they construct a commutative algebra $\mathcal{A}_x$ of difference operators in $\mathbf{R}^p$, depending on $p+3$ parameters, which is diagonalized by the multivariable Racah polynomials considered by Tratnik. In the particular case $p=2$, the bivariate Racah polynomials are defined in terms of univariate Racah polynomials as

$$R_{n,m}(s,t;\beta_0,\beta_1,\beta_2,\beta_3,N) = r_n(\beta_1-\beta_0-1,\beta_2-\beta_1-1,-t-1,\beta_1+t;s) \\ \times r_m(2n+\beta_2-\beta_0-1,\beta_3-\beta_2-1,n-N-1,n+\beta_2+N;t-n),$$

which are polynomials in the lattices $x(s)=s(s+\beta_1)$ and $y(t)=t(t+\beta_2)$. These polynomials coincide with the bivariate Racah polynomials of parameters a_1, a_2, a_3, γ, and η introduced by Tratnik after the substitutions

$$\beta_0=a_1-\eta-1,\;\; \beta_1=a_1,\;\; \beta_2=a_1+a_2,\;\; \beta_3=a_1+a_2+a_3,\;\; N=-\gamma-1.$$

Let us a look to some highlights in the nice paper of Geronimo and Iliev.

Theorem

The bivariate Racah polynomials are solution of the following fourth-order linear partial divided-difference equation

$$
\begin{aligned}
&f_1(x(s),y(t))\mathbb{D}_x^2\mathbb{D}_y^2 R_{n,m}(s,t) + f_2(x(s),y(t))\mathbb{S}_x\mathbb{D}_x\mathbb{D}_y^2 R_{n,m}(s,t)\\
&+ f_3(x(s),y(t))\mathbb{S}_y\mathbb{D}_y\mathbb{D}_x^2 R_{n,m}(s,t) + f_4(x(s),y(t))\mathbb{S}_x\mathbb{D}_x\mathbb{S}_y\mathbb{D}_y R_{n,m}(s,t)\\
&+ f_5(x(s))\mathbb{D}_x^2 R_{n,m}(s,t) + f_6(y(t))\mathbb{D}_y^2 R_{n,m}(s,t) + f_7(x(s))\mathbb{S}_x\mathbb{D}_x R_{n,m}(s,t)\\
&+ f_8(y(t))\mathbb{S}_y\mathbb{D}_y R_{n,m}(s,t) + (m+n)(\beta_3-\beta_0+m+n-1)R_{n,m}(s,t) = 0,
\end{aligned}
$$

where $R_{n,m}(s,t) := R_{n,m}(s,t;\beta_0,\beta_1,\beta_2,\beta_3,N)$, *and the coefficients* f_i, $i = 1,\ldots,8$ *are polynomials in the lattices* $x(s)$ *and* $y(t)$ *given by*

$$
\begin{aligned}
f_8(y(t)) &= (\beta_0 - \beta_3)y(t) - N(\beta_0 - \beta_2)(\beta_3 + N), \\
f_7(x(s)) &= (\beta_0 - \beta_3)x(s) - N(\beta_0 - \beta_1)(\beta_3 + N), \\
f_6(y(t)) &= -(y(t))^2 + \frac{1}{2}(2N^2 + 2\beta_3(\beta_0 + N) \\
&\quad - \beta_2(\beta_3 + \beta_0))y(t) - \frac{1}{2}N\beta_2(\beta_0 - \beta_2)(\beta_3 + N), \\
f_5(x(s)) &= -(x(s))^2 + \frac{1}{2}(2\beta_3(N + \beta_0) \\
&\quad + 2N^2 - \beta_1(\beta_3 + \beta_0))x(s) \\
&\quad - \frac{1}{2}N\beta_1(\beta_0 - \beta_1)(\beta_3 + N), \\
f_4(x(s), y(t)) &= -2x(s)y(t) \\
&\quad + (2N^2 + \beta_2(1 - \beta_0) + \beta_3(\beta_0 - 1 + 2N))x(s) \\
&\quad + (\beta_0 - \beta_1)(\beta_3 + 1)y(t) - N(\beta_0 - \beta_1)(\beta_2 + 1)(\beta_3 + N), \\
&[\dots]
\end{aligned}
$$

Theorem

The polynomial $R_{n,m}^{(1,0)}(s,t) := \mathbb{D}_x R_{n,m}(s,t)$ *is solution of the following* *fourth-order* *linear partial divided-difference equation*

$$\begin{aligned}
&f_{11}(x(s),y(t))\mathbb{D}_x^2\mathbb{D}_y^2 R_{n,m}^{(1,0)}(s,t) + f_{21}(x(s),y(t))\mathbb{S}_x\mathbb{D}_x\mathbb{D}_y^2 R_{n,m}^{(1,0)}(s,t)\\
&+ f_{31}(x(s),y(t))\mathbb{S}_y\mathbb{D}_y\mathbb{D}_x^2 R_{n,m}^{(1,0)}(s,t) + f_{41}(x(s),y(t))\mathbb{S}_x\mathbb{D}_x\mathbb{S}_y\mathbb{D}_y R_{n,m}^{(1,0)}(s,t)\\
&+ f_{51}(x(s))\mathbb{D}_x^2 R_{n,m}^{(1,0)}(s,t) + f_{61}(y(t))\mathbb{D}_y^2 R_{n,m}^{(1,0)}(s,t)\\
&+ f_{71}(x(s))\mathbb{S}_x\mathbb{D}_x R_{n,m}^{(1,0)}(s,t) + f_{81}(y(t))\mathbb{S}_y\mathbb{D}_y R_{n,m}^{(1,0)}(s,t)\\
&+ (m+n-1)(\beta_3-\beta_0+m+n) R_{n,m}^{(1,0)}(s,t) = 0,
\end{aligned}$$

where the coefficients $f_{i1},\ i=1,\dots,8$ *are polynomials in the lattices* $x(s)$ *and* $y(t)$

Theorem

The polynomial $R^{(0,1)}_{n,m}(s,t) := \mathbb{D}_y R_{n,m}(x(s),y(t))$ *is solution of the following* *fourth-order* *linear partial divided-difference equation*

$$
\begin{aligned}
&f_{12}(x(s),y(t))\mathbb{D}_x^2\mathbb{D}_y^2 R^{(0,1)}_{n,m}(s,t) + f_{22}(x(s),y(t))\mathbb{S}_x\mathbb{D}_x\mathbb{D}_y^2 R^{(0,1)}_{n,m}(s,t)\\
&+ f_{32}(x(s),y(t))\mathbb{S}_y\mathbb{D}_y\mathbb{D}_x^2 R^{(0,1)}_{n,m}(s,t) + f_{42}(x(s),y(t))\mathbb{S}_x\mathbb{D}_x\mathbb{S}_y\mathbb{D}_y R^{(0,1)}_{n,m}(s,t)\\
&+ f_{52}(x(s))\mathbb{D}_x^2 R^{(0,1)}_{n,m}(s,t) + f_{62}(y(t))\mathbb{D}_y^2 R^{(0,1)}_{n,m}(s,t)\\
&+ f_{72}(x(s))\mathbb{S}_x\mathbb{D}_x R^{(0,1)}_{n,m}(s,t) + f_{82}(y(t))\mathbb{S}_y\mathbb{D}_y R^{(0,1)}_{n,m}(s,t)\\
&+ (m+n-1)(\beta_3-\beta_0+m+n) R^{(0,1)}_{n,m}(s,t) = 0,
\end{aligned}
$$

where the coefficients f_{i2}, $i = 1,\ldots,8$ *are polynomials in the lattices* $x(s)$ *and* $y(t)$.

Notice that for $i = 1, \ldots, 8$ we have

$$f_{i1}(x(s), y(t)) \\ = f_i(x(s-1/2), y(t-1); \beta_0, 1+\beta_1, 2+\beta_2, 2+\beta_3, N-1),$$

and

$$f_{i2}(x(s), y(t)) = f_i(x(s), y(t-1/2); \beta_0, \beta_1, \beta_2+1, \beta_3+2, N-1),$$

where $f_i = f_i(x(s), y(t); \beta_0, \beta_1, \beta_2, \beta_3, N)$ can be explicitly given. As a consequence of the latter equalities, we obtain the following relation

$$\mathbb{D}_x R_{n,m}(s, t; \beta_0, \beta_1, \beta_2, \beta_3, N) = n(n-\beta_0+\beta_2-1) \\ \times R_{n-1,m}(s-1/2, t-1; \beta_0, \beta_1+1, \beta_2+2, \beta_3+2, N-1).$$

We have recently proposed a conjecture about the shape of the form of the hypergeometric-type divided-difference equation satisfied by the p-variate Racah polynomials.

q-quadratic lattices!

Bivariate Askey-Wilson polynomials defined by

$$\begin{aligned} P_{n,m}(s,t;a,b,c,d,e_2|q) &= p_n(x(s);a,b,e_2q^t,e_2q^{-t}|q) \\ &\times p_m(y(t);ae_2q^n,be_2q^n,c,d|q), \\ \tilde{P}_{n,m}(s,t;a,b,c,d,e_2|q) &= p_n(x(s);ce_2q^m,de_2q^m,a,b|q) \\ &\times p_m(y(t);c,d,e_2q^s,e_2q^{-s}|q), \end{aligned}$$

have been introduced by Gasper and Rahman in 2005. They are polynomials of total degree $n+m$ in the variables $x(s)$ and $y(t) := x(t)$. The univariate Askey-Wilson polynomials are defined by

$$\begin{aligned} & p_n(x(s);a,b,c,d|q) \\ & \quad = \frac{(ab,ac,ad;q)_n}{a^n} {}_4\phi_3\left(\begin{matrix} q^{-n}, abcdq^{n-1}, aq^s, aq^{-s} \\ ab, ac, ad \end{matrix} \middle| q;q \right); \end{aligned}$$

they are polynomials of degree n in the q-quadratic lattice

$$x(s) = \cos\theta = \frac{q^s+q^{-s}}{2}, \qquad q^s = e^{i\theta}.$$

Theorem

Let

$$x(s) = \frac{q^s + q^{-s}}{2} = \cos\theta_1, \quad y(t) = \frac{q^t + q^{-t}}{2} = \cos\theta_2.$$

The bivariate Askey-Wilson polynomials are solution of the following fourth-order linear partial divided-difference equation

$$\begin{aligned} & f_1(x(s), y(t))\mathbb{D}_x^2\mathbb{D}_y^2 P_{n,m}(s,t) + f_2(x(s), y(t))\mathbb{S}_x\mathbb{D}_x\mathbb{D}_y^2 P_{n,m}(s,t) \\ & + f_3(x(s), y(t))\mathbb{S}_y\mathbb{D}_y\mathbb{D}_x^2 P_{n,m}(s,t) + f_4(x(s), y(t))\mathbb{S}_x\mathbb{D}_x\mathbb{S}_y\mathbb{D}_y P_{n,m}(s,t) \\ & + f_5(x(s))\mathbb{D}_x^2 P_{n,m}(s,t) + f_6(y(t))\mathbb{D}_y^2 P_{n,m}(s,t) + f_7(x(s))\mathbb{S}_x\mathbb{D}_x P_{n,m}(s,t) \\ & + f_8(y(t))\mathbb{S}_y\mathbb{D}_y P_{n,m}(s,t) + \lambda_{n,m} P_{n,m}(s,t) = 0, \end{aligned}$$

where $\lambda_{n,m} = 16q^{-(m+n)+3}(1 - q^{m+n})(1 - abcde_2^2 q^{m+n-1})$*,* $P_{n,m}(s,t)$ *stands for* $P_{n,m}(s,t;a,b,c,d,e_2|q)$ *or* $\tilde{P}_{n,m}(s,t;a,b,c,d,e_2|q)$*,*

$$
\begin{aligned}
f_8(y(t)) &= 8q^2(q-1)\big(2(1-abcde_2^2)y(t)+(d+c)(abe_2^2-1)\\
&\quad +(cd-1)(a+b)e_2\big),\\
f_7(x(s)) &= 8q^2(q-1)\big(2(1-abcde_2^2)x(s)+(cde_2^2-1)(a+b)\\
&\quad +(d+c)(ab-1)e_2\big),\\
f_6(y(t)) &= 4q^{3/2}(q-1)^2\Big(-2(abcde_2^2+1)(y(t))^2\\
&\quad +\Big((d+c)(abe_2^2+1)+(cd+1)(a+b)e_2\Big)y(t)\\
&\quad +(cd-1)(abe_2^2-1)-(d+c)(a+b)e_2\Big),\\
f_5(x(s)) &= 4q^{3/2}(q-1)^2\Big(-2(abcde_2^2+1)(x(s))^2\\
&\quad +\Big((cde_2^2+1)(a+b)+(d+c)(ab+1)e_2\Big)x(s)\\
&\quad +(cde_2^2-1)(ab-1)-(d+c)(a+b)e_2\Big),\\
&\quad [\ldots]
\end{aligned}
$$

Proposition

The difference derivative of the bivariate Askey-Wilson polynomials

$$\mathbb{D}_x P_{n,m}(s,t;a,b,c,d,e_2|q) := P_{n,m}^{(1,0)}(s,t)$$

are solution of a fourth-order linear partial divided-difference equation

$$\begin{gathered} f_{11}(x(s),y(t))\mathbb{D}_x^2\mathbb{D}_y^2 P_{n,m}^{(1,0)}(s,t) + f_{12}(x(s),y(t))\mathbb{S}_x\mathbb{D}_x\mathbb{D}_y^2 P_{n,m}^{(1,0)}(s,t) \\ + f_{13}(x(s),y(t))\mathbb{S}_y\mathbb{D}_y\mathbb{D}_x^2 P_{n,m}^{(1,0)}(s,t) + f_{14}(x(s),y(t))\mathbb{S}_x\mathbb{D}_x\mathbb{S}_y\mathbb{D}_y P_{n,m}^{(1,0)}(s,t) \\ + f_{15}(x(s))\mathbb{D}_x^2 P_{n,m}^{(1,0)}(s,t) + f_{16}(y(t))\mathbb{D}_y^2 P_{n,m}^{(1,0)}(s,t) \\ + f_{17}(x(s))\mathbb{S}_x\mathbb{D}_x P_{n,m}^{(1,0)}(s,t) + f_{18}(y(t))\mathbb{S}_y\mathbb{D}_y P_{n,m}^{(1,0)}(s,t) \\ + 16\, q^{-m-n+4}\left(q^{m+n-1}-1\right)\left(abcde_2^2 q^{m+n}-1\right) P_{n,m}^{(1,0)}(s,t) = 0, \end{gathered}$$

with

$$f_{1i}(x(s),y(t)) = f_i(s,t;aq^{\frac{1}{2}},bq^{\frac{1}{2}},c,d,e_2 q^{\frac{1}{2}}),\ i=1,\ldots,8,$$

Proposition

The difference derivative of the bivariate Askey-Wilson polynomials

$$\mathbb{D}_y P_{n,m}(s,t;a,b,c,d,e_2|q) := P_{n,m}^{(0,1)}(s,t)$$

are solution of a fourth-order linear partial divided-difference equation

$$\begin{aligned}
& f_{21}(x(s),y(t))\mathbb{D}_x^2\mathbb{D}_y^2 P_{n,m}^{(0,1)}(s,t) + f_{22}(x(s),y(t))\mathbb{S}_x\mathbb{D}_x\mathbb{D}_y^2 P_{n,m}^{(0,1)}(s,t) \\
& + f_{23}(x(s),y(t))\mathbb{S}_y\mathbb{D}_y\mathbb{D}_x^2 P_{n,m}^{(0,1)}(s,t) + f_{24}(x(s),y(t))\mathbb{S}_x\mathbb{D}_x\mathbb{S}_y\mathbb{D}_y P_{n,m}^{(0,1)}(s,t) \\
& + f_{25}(x(s))\mathbb{D}_x^2 P_{n,m}^{(0,1)}(s,t) + f_{26}(y(t))\mathbb{D}_y^2 P_{n,m}^{(0,1)}(s,t) \\
& + f_{27}(x(s))\mathbb{S}_x\mathbb{D}_x P_{n,m}^{(0,1)}(s,t) + f_{28}(y(t))\mathbb{S}_y\mathbb{D}_y P_{n,m}^{(0,1)}(s,t) \\
& + 16\, q^{4-n-m}\left(q^{m+n-1}-1\right)\left(abcde_2^2 q^{m+n}-1\right) P_{n,m}^{(0,1)}(s,t) = 0,
\end{aligned}$$

with

$$f_{2i}(x(s),y(t)) = f_i(s,t;a,b,cq^{\frac{1}{2}},dq^{\frac{1}{2}},e_2q^{\frac{1}{2}}), \quad i=1,\ldots,8.$$

By using the same ideas as for the continuous case, discrete case and their q-analogues we have succeeded to obtain the explicit forms of the matrices appearing in the three-term recurrence relation satisfied by all the known families of bivariate orthogonal polynomials on nonuniform lattices.

We denote by

$$\mathbf{x}^n = \left(x(s)^n, x(s)^{n-1}y(t), \cdots, x(s)y(t)^{n-1}, y(t)^n\right)^{\mathsf{T}},$$

the column vector of the monomials $x(s)^{n-k}y(t)^k$. Let $\mathbf{P}_n$ denote the (column) polynomial vector of the polynomials $P_{n-k,k}(x(s), y(t))$ of total degree n,

$$\mathbf{P}_n = (P_{n,0}(x(s), y(t)), P_{n-1,1}(x(s), y(t)), \ldots, P_{0,n}(x(s), y(t)))^{\mathsf{T}}.$$

Then,

$$\mathbf{P}_n = G_{n,n}\mathbf{x}^n + G_{n,n-1}\mathbf{x}^{n-1} + G_{n,n-2}\mathbf{x}^{n-2} + \cdots + G_{n,0}\,\mathbf{x}^0, \qquad (1)$$

where $G_{n,j}$ are matrices of size $(n+1) \times (j+1)$ and $G_{n,n}$ is a nonsingular square matrix of size $(n+1) \times (n+1)$.

We have the relations

$$x(s)\mathbf{x}^n = L_{n,1}\mathbf{x}^{n+1}, \; y(t)\mathbf{x}^n = L_{n,2}\mathbf{x}^{n+1}, \tag{2}$$

where $L_{n,1}$ and $L_{n,2}$ are $(n+1)\times(n+2)$ size matrices given by

$$L_{n,1} = \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix}, \qquad L_{n,2} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & 1 \end{pmatrix}.$$

Theorem

The polynomials $\mathbf{P}_n$ *are solution of the three-term recurrence relations*

$$x_j \mathbf{P}_n = A_{n,j} \mathbf{P}_{n+1} + B_{n,j} \mathbf{P}_n + C_{n,j} \mathbf{P}_{n-1}, \quad j = 1, 2, \tag{3}$$

with the initial conditions $\mathbf{P}_{-1} = 0$ *and* $\mathbf{P}_0 = 1$*, where* $x_1 = x(s)$ *and* $x_2 = y(t)$ *and the matrices* $A_{n,j}$ *of size* $(n+1) \times (n+2)$*,* $B_{n,j}$ *of size* $(n+1) \times (n+1)$*, and* $C_{n,j}$ *of size* $(n+1) \times n$ *are given by*

$$\begin{aligned}
A_{n,j} &= G_{n,n} L_{n,j} G^{-1}_{n+1,n+1}, \quad n \geq 0, \\
B_{0,j} &= \left(-A_{0,j} G_{1,0}\right) G^{-1}_{0,0}, \\
B_{n,j} &= \left(G_{n,n-1} L_{n-1,j} - A_{n,j} G_{n+1,n}\right) G^{-1}_{n,n}, \quad n \geq 1, \\
C_{1,j} &= \left(-A_{1,j} G_{2,0} - B_{1,j} G_{1,0}\right) G^{-1}_{0,0}, \\
C_{n,j} &= \left(G_{n,n-2} L_{n-2,j} - A_{n,j} G_{n+1,n-1} - B_{n,j} G_{n,n-1}\right) G^{-1}_{n-1,n-1}, \quad n \geq 2.
\end{aligned}$$

It follows clearly that to compute the coefficients $A_{n,j}$, $B_{n,j}$ and $C_{n,j}$, we need to know explicitly the coefficients $G_{n,n}$, $G_{n,n-1}$, $G_{n,n-2}$ of $\mathbf{P}_n$.

From the definition of the bivariate Askey-Wilson polynomials, we get

$$P_{n,m}(s,t;a,b,c,d,e_2|q) =$$
$$C(n,m)\left(x(s)^n y(t)^m,\ x(s)^{n-1}y(t)^{m+1},\dots,x(s)y(t)^{m+n-1},y(t)^{m+n}\right)^{\mathsf{T}} + \cdots,$$

where

$$C(n,m) = (A_i(n,m))_{0\le i\le n},$$

is the leading coefficient vector of $P_{n,m}(s,t;a,b,c,d,e_2|q)$ in the basis $x(s)^i y(t)^j$ with

$$A_i(n,m) = (-2\,a)^{n+m}\, q^{\frac{1}{2}(n+m)(m+n-1)} {e_2}^{i+m} C_2(n,m,m) C_1(n,n-i),$$

for $i = 0, 1, \dots, n$, where

$$C_1(n,i) = \frac{(ab;q)_n\left(q^{-n}, abe_2^2q^{n-1};q\right)_i q^i}{(ab,q;q)_i\, a^n},$$

$$C_2(n,m,j) = \frac{\left(abe_2^2q^{2\,n}, ace_2q^n, ade_2q^n;q\right)_m \left(q^{-m}, abcde_2^2q^{2\,n+m-1};q\right)_j q^j}{\left(abe_2^2q^{2\,n}, ace_2q^n, ade_2q^n, q;q\right)_j (ae_2q^n)^m}.$$

For the second bivariate Askey-Wilson polynomial family, we have

$$\tilde{P}_{n,m}(s,t;a,b,c,d,e_2|q) =$$
$$\tilde{C}(n,m)\left(x(s)^n y(t)^m, x(s)^{n+1}y(t)^{m-1},\ldots,x(s)^{n+m-1}y(t),x(s)^{n+m}\right)^{\mathrm{T}} + \cdots,$$

where

$$\tilde{C}(n,m) = (\tilde{A}_j(n,m))_{0\leq j\leq m}$$

and

$$\tilde{A}_j(n,m) = (-2\,c)^{m+n}\,q^{1/2\,(m+n)(m+n-1)}\,{e_2}^{n+j}\,\tilde{C}_1(n,m,n)\tilde{C}_2(m,m-j),$$

where

$$\tilde{C}_1(n,m,i) = \frac{\left(cde_2^2q^{2\,m},ace_2q^m,bce_2q^m;q\right)_n\left(q^{-n},abcde_2^2q^{2\,m+n-1};q\right)_i q^i}{\left(cd{e_2}^2q^{2\,m},ace_2q^m,bce_2q^m,q;q\right)_i\left(ce_2q^m\right)^n},$$
$$\tilde{C}_2(m,j) = \frac{(cd;q)_m\left(q^{-m},cde_2^2q^{m-1};q\right)_j q^j}{c^m\,(cd,q;q)_j}.$$

Since

$$P_{n-j,j}(x(s),y(t)) = \sum_{i=0}^{n-j} A_i(n-j,j)x(s)^{n-j-i}y(t)^{i+j} + \cdots ,$$

and

$$\tilde{P}_{n-j,j}(x(s),y(t)) = \sum_{k=0}^{j} \tilde{A}_k(n-j,j)x(s)^{n-j+k}y(t)^{j-k} + \cdots ,$$

it follows that $G_{n,n} = (g_{k,l}(n))_{0\le k,l\le n}$ is an upper triangular matrix where

$$\begin{aligned} g_{k,l}(n) = A_{l-k}(n-k,k) &= (-2)^n q^{\frac{1}{2}n^2+\frac{1}{2}(-2k+1)n+k^2+k-l} {e_2}^{l-k} \\ &\times \frac{(ab;q)_{n-k}\,(q^{k-n};q)_{n-l}}{(q;q)_{n-l}\,(q;q)_k} \\ &\times \frac{(abe_2{}^2q^{n-k-1};q)_{n-l}\,(q^{-k};q)_k\,(abcde_2{}^2q^{2n-k-1};q)_k}{(ab;q)_{n-l}}, \end{aligned}$$

and $\tilde{G}_{n,n} = (\tilde{g}_{k,l}(n))_{0 \le k,l \le n}$ is a lower triangular matrix where

$$\tilde{g}_{k,l}(n) = A_{k-l}(n-k,k) = (-2)^n \, q^{\frac{1}{2}n^2+\frac{1}{2}(-2\,k+1)n+k^2-k+l} {e_2}^{k-l} \times \frac{\left(q^{-n+k}, abcd{e_2}^2 q^{k+n-1}; q\right)_{n-k} (dc;q)_k \left(q^{-k}, cd{e_2}^2 q^{k-1}; q\right)_l}{(q;q)_{n-k} (dc;q)_l (q;q)_l},$$

where we consider

$$\tilde{\mathbf{P}}_n = (\tilde{P}_{n-k,k}(x(s),y(t)))_{k=0,\ldots,n}^{\mathrm{T}} = \tilde{G}_{n,n}\mathbf{x}^n + \tilde{G}_{n,n-1}\mathbf{x}^{n-1} + \cdots .$$

Connection!

Let $\hat{\mathbf{P}}_n$ be the column vector of monic bivariate Askey-Wilson polynomials. Then, we have

$$\mathbf{P}_n = G_{n,n}\,\hat{\mathbf{P}}_n, \qquad \tilde{\mathbf{P}}_n = \tilde{G}_{n,n}\,\hat{\mathbf{P}}_n.$$

As a consequence, we obtain the following connection formulae between the two families of bivariate Askey-Wilson polynomials

$$\mathbf{P}_n = G_{n,n}(\tilde{G}_{n,n})^{-1}\,\tilde{\mathbf{P}}_n,\; n \geq 0.$$

It is therefore sufficient to get the recurrence relation for $\mathbf{P}_n$ since the one of $\tilde{\mathbf{P}}_n$ follows from this connection formula. This connection formula can also be written as

$$P_{n-j,j}(x(s),y(t)) = \sum_{l=0}^{n} b_{j,l}\tilde{P}_{n-l,l}(x(s),y(t)),\; j = 0,1,\ldots,n,$$

where $(b_{j,0}, b_{j,1}, \ldots, b_{j,n})$ is the j^{th} row of the matrix $G_{n,n}(\tilde{G}_{n,n})^{-1}$.

To compute the coefficients $G_{n,n-1}$ and $G_{n,n-2}$, we use the fourth-order linear partial divided-difference equations satisfied by the bivariate Askey-Wilson polynomials and the following results.
The action of the operators $\mathbb{D}_x$ and $\mathbb{S}_x$ on $x(s)^n$ is given for $i = 1, 2$ by

$$\mathbb{D}_x x(s)^n = H_{n,n-1,i} x(s)^{n-1} + H_{n,n-2,i} x(s)^{n-2} + H_{n,n-3,i} x(s)^{n-3} + \cdots ,$$
$$\mathbb{S}_x x(s)^n = Q_{n,n,i} x(s)^n + Q_{n,n-1,i} x(s)^{n-1} + Q_{n,n-2,i} x(s)^{n-2} + \cdots ,$$

where

$$H_{n,n-1,i} = \frac{\sqrt{q}}{q-1}(q^{\frac{n}{2}} - q^{-\frac{n}{2}}),\ H_{n,n-2,i} = 0,$$
$$H_{n,n-3,i} = \frac{(-2+n)\left(q^{1/2-n/2} - q^{1/2+n/2}\right) + n\left(q^{-1/2+n/2} - q^{3/2-n/2}\right)}{4\,q-4},$$
$$Q_{n,n,i} = \frac{1}{2}\Big(q^{n/2} + q^{-n/2}\Big),$$
$$Q_{n,n-1,i} = 0,\ Q_{n,n-2,i} = -\frac{1}{8}\,nq^{-\frac{n}{2}}(q-1)(q^{n-1}-1).$$

It follows that

$$\mathbb{D}_x \mathbf{x}^n = E_{n,n-1,1}\mathbf{x}^{n-1} + E_{n,n-2,1}\mathbf{x}^{n-2} + E_{n,n-3,1}\mathbf{x}^{n-3} + \cdots ,$$
$$\mathbb{S}_x \mathbf{x}^n = M_{n,n,1}\mathbf{x}^{n} + M_{n,n-1,1}\mathbf{x}^{n-1} + M_{n,n-2,1}\mathbf{x}^{n-2} + \cdots ,$$
$$\mathbb{D}_y \mathbf{x}^n = E_{n,n-1,2}\mathbf{x}^{n-1} + E_{n,n-2,2}\mathbf{x}^{n-2} + E_{n,n-3,2}\mathbf{x}^{n-3} + \cdots ,$$
$$\mathbb{S}_y \mathbf{x}^n = M_{n,n,2}\mathbf{x}^{n} + M_{n,n-1,2}\mathbf{x}^{n-1} + M_{n,n-2,2}\mathbf{x}^{n-2} + \cdots ,$$

where $\mathbf{E}_{n,j,i} = (p_{n,j,k,l,i})_{1 \leq k \leq n+1, 1 \leq l \leq j+1}$, $i = 1, 2$, ($j = n-1, n-2, n-3$) are matrices of size $(n+1) \times (j+1)$ and $\mathbf{M}_{n,j,i} = (r_{n,j,k,l,i})_{1 \leq k \leq n+1, 1 \leq l \leq j+1}$, $i = 1, 2$, ($j = n, n-1, n-2$) are matrices of size $(n+1) \times (j+1)$ given by

$$p_{n,j,k,l,1} = \begin{cases} H_{n-k+1,j-k+1,1}, & k = l \\ 0, & k \neq l, \end{cases} , \quad p_{n,j,k,l,2} = \begin{cases} H_{n-j+l-1,l-1,2}, & k = l+n-j \\ 0, & k \neq l+n-j, \end{cases}$$
$$r_{n,j,k,l,1} = \begin{cases} Q_{n-k+1,j-k+1,1}, & k = l \\ 0, & k \neq l, \end{cases} , \quad r_{n,j,k,l,2} = \begin{cases} Q_{n-j+l-1,l-1,2}, & k = l+n-j \\ 0, & k \neq l+n-j \end{cases} .$$

We substitute the expansion in the divided-difference equation, we first use $\mathbb{D}_x \mathbf{x}^n$, $\mathbb{S}_x \mathbf{x}^n$, $\mathbb{D}_y \mathbf{x}^n$, $\mathbb{S}_y \mathbf{x}^n$ and then $x(s)\mathbf{x}^n$, $y(t)\mathbf{x}^n$. By equating the coefficients of $\mathbf{x}^{n-1}$ and $\mathbf{x}^{n-2}$ we obtain the explicit expressions of the matrices $G_{n,n-1}$ and $G_{n,n-2}$ in terms of the nonsingular matrix $G_{n,n}$,

$$\begin{cases} G_{n,n-1} &= \dfrac{1}{\lambda_{n-1} - \lambda_n} G_{n,n} \mathbf{S}_n, \\ G_{n,n-2} &= \dfrac{1}{\lambda_{n-2} - \lambda_n} \left(G_{n,n} \mathbf{T}_n + G_{n,n-1} \mathbf{S}_{n-1} \right), \end{cases} \tag{4}$$

where $\lambda_n = 16 q^{3-n} (1 - q^n) \left(1 - abcde_2^2 q^{n-1}\right)$.

The matrix $\mathbf{S}_n$ of size $(n+1)\times n$ is given by

$$\mathbf{S}_n = \begin{pmatrix} s_{1,1} & 0 & \dots & 0 \\ s_{2,1} & s_{2,2} & \ddots & \vdots \\ 0 & s_{3,2} & \ddots & 0 \\ \vdots & \ddots & \ddots & s_{n,n} \\ 0 & \dots & 0 & s_{n+1,n} \end{pmatrix} \qquad (n\geq 1),$$

where for $k=1,\dots,n$,

$$s_{k,k} = 8\,q^{1-n}\left(q^{n-k+1}-1\right)$$
$$\Big((a+b)\,(cd{e_2}^2q^{n+k}-q^2)+q\,(c+d)\,(q^n ab-q^k)\,e_2\Big),$$
$$s_{k+1,k} = 8\,\left(q^k-1\right)q^{-n-k+2}$$
$$\Big((c+d)\,(ab{e_2}^2q^{2\,n}-q^{k+1})+e_2\,q^n\,(a+b)\,(cdq^k-q)\Big).$$

The matrix $\mathbf{T}_n$ of size $(n+1)\times(n-1)$ is given by

$$\mathbf{T}_n = \begin{pmatrix} t_{1,1} & 0 & \cdots & \cdots & 0 \\ t_{2,1} & t_{2,2} & \ddots & & \vdots \\ t_{3,1} & t_{3,2} & \ddots & \ddots & \vdots \\ 0 & t_{4,2} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & t_{n-1,n-1} \\ \vdots & & \ddots & \ddots & t_{n,n-1} \\ 0 & \cdots & \cdots & 0 & t_{n+1,n-1} \end{pmatrix} \qquad (n \geq 2),$$

where, for $1 \leq k \leq n-1$,

$$t_{k,k} = q^{-2k-n}\left(4(q+1)q^{k+n}\left((ab+q)(cde_2^2q^{2k}+q^3)+e_2(a+b)(c+d)q^{k+2}\right)\right.$$
$$+4q^{2n}\left(-cde_2^2q^{2k}\left(ab(q+1)(k(q-1)+n(-q)+n+1)+q^2\right)\right.$$
$$\left.-e_2(a+b)(c+d)q^{k+3}-abq^4\right)-4e_2(a+b)(c+d)q^{3k+2}-4q^{2k+3}$$
$$\left.\left(ab+q^2(-k+n+1)+k-n+q\right)-4cde_2^2q^{4k+1}\right),$$

and we can also give the expression for $t_{k+1,k}$.

More connection

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Thank you