Multiple Orthogonal Polynomials

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Workshop "Introduction to Orthogonal Polynomials and Applications"

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- Three term recurrence relation:

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- Classical orthogonal polynomials: Jacobi - Laguerre - Hermite

Plan of the talk

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- Definitions

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We use **multi-indices** $\vec{n} = (n_1, n_2, \ldots, n_r) \in \mathbb{N}^r$ and denote their **length** by $|\vec{n}| = n_1 + n_2 + \cdots + n_r$.

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Definition (type I)

Type I multiple orthogonal polynomials for $\vec{n}$ consist of the vector $(A_{\vec{n},1}, \ldots, A_{\vec{n},r})$ of r polynomials, with $\deg A_{\vec{n},j} \leq n_j - 1$, for which

$$\int x^k \sum_{j=1}^{r} A_{\vec{n},j}(x)\, d\mu_j(x) = 0, \qquad 0 \leq k \leq |\vec{n}| - 2,$$

with normalization

$$\int x^{|\vec{n}|-1} \sum_{j=1}^{r} A_{\vec{n},j}(x)\, d\mu_j(x) = 1.$$

Definition: type II MOPS

Definition (type II)

The type II multiple orthogonal polynomial for $\vec{n}$ is the **monic** polynomial $P_{\vec{n}}$ of degree $|\vec{n}|$ for which

$$\int x^k P_{\vec{n}}(x)\, d\mu_j(x) = 0, \qquad 0 \le k \le n_j - 1,$$

for $1 \le j \le r$.

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a multi-index $\vec{n}$ is **normal** if the type I vector $(A_{\vec{n},1}, \ldots A_{\vec{n},r})$ exists and is unique

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$$\det \begin{pmatrix} M_{n_1}^{(1)} \\ M_{n_2}^{(2)} \\ \vdots \\ M_{n_r}^{(r)} \end{pmatrix} \neq 0, \qquad M_{n_j}^{(j)} = \begin{pmatrix} m_0^{(j)} & m_1^{(j)} & \cdots & m_{|\vec{n}|-1}^{(j)} \\ m_1^{(j)} & m_2^{(j)} & \cdots & m_{|\vec{n}|}^{(j)} \\ \vdots & \vdots & \cdots & \vdots \\ m_{n_j-1}^{(j)} & m_{n_j}^{(j)} & \cdots & m_{|\vec{n}|+n_j-2}^{(j)} \end{pmatrix},$$

$$m_k^{(j)} = \int x^k \, d\mu_j(x).$$

Special systems: Angelesco systems

Definition (Angelesco system)

The measures $(\mu_1, \ldots, \mu_r)$ are an **Angelesco system** if the supports of the measures are subsets of disjoint intervals Δ_j, i.e., $\text{supp}(\mu_j) \subset \Delta_j$ and $\Delta_i \cap \Delta_j = \emptyset$ whenever $i \neq j$.

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Usually one allows that the intervals are touching, i.e., $\mathring{\Delta}_i \cap \mathring{\Delta}_j = \emptyset$ whenever $i \neq j$.

Special systems: Angelesco systems

Theorem (Angelesco, Nikishin)

The type II multiple orthogonal polynomial $P_{\vec{n}}$ has exactly n_j distinct zeros on $\mathring{\Delta}_j$ for $1 \leq j \leq r$.

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Corollary

Every multi-index $\vec{n}$ is normal (an Angelesco system is perfect).

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Exercise

Show that every $A_{\vec{n},j}$ has $n_j - 1$ zeros on $\overset{\circ}{\Delta}_j$.

Special systems: AT systems

Definition

The functions $\varphi_1, \ldots, \varphi_n$ are a **Chebyshev system** on $[a, b]$ if every linear combination $\sum_{i=1}^n a_i \varphi_i$ with $(a_1, \ldots, a_n) \neq (0, \ldots, 0)$ has at most $n-1$ zeros on $[a, b]$.

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Definition (AT-system)

The measures $(\mu_1, \ldots, \mu_r)$ are an **AT-system** on the interval $[a, b]$ if the measures are all absolutely continuous with respect to a positive measure μ on $[a, b]$, i.e., $d\mu_j(x) = w_j(x)\, d\mu(x)$ $(1 \leq j \leq r)$, and for every $\vec{n}$ the functions

$$w_1(x), xw_1(x), \ldots, x^{n_1-1}w_1(x),\ w_2(x), xw_2(x), \ldots, x^{n_2-1}w_2(x), \\ \ldots, w_r(x), xw_r(x), \ldots, x^{n_r-1}w_r(x)$$

are a Chebyshev system on $[a, b]$.

Special systems: AT-systems

Theorem

For an AT-system the function

$$Q_{\vec{n}}(x) = \sum_{j=1}^{r} A_{\vec{n},j}(x) w_j(x)$$

has exactly $|\vec{n}| - 1$ *sign changes on* (a, b).

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Theorem

If $(\mu_1, \ldots, \mu_r)$ *is an AT-system, then the type II multiple orthogonal polynomial* $P_{\vec{n}}$ *has exactly* $|\vec{n}|$ *distinct zeros on* (a, b).

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Every multi-index in an AT-system is normal (an AT-system is perfect).

Special systems: Nikishin systems

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Definition (Nikishin system for $r = 2$)

A Nikishin system of order $r = 2$ consists of two measures (μ_1, μ_2), both supported on an interval Δ_2, and such that

$$\frac{d\mu_2(x)}{d\mu_1(x)} = \int_{\Delta_1} \frac{d\sigma(t)}{x - t},$$

where σ is a positive measure on an interval Δ_1 and $\Delta_1 \cap \Delta_2 = \emptyset$.

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Theorem (Nikishin, Driver-Stahl)

A Nikishin system of order two is perfect.

Special systems: Nikishin systems

Notation: $\langle \sigma_1, \sigma_2 \rangle$ is a measure which is absolutely continuous with respect to σ_1 and for which the Radon-Nikodym derivative is a Stieltjes transform of σ_2:

$$d\langle \sigma_1, \sigma_2 \rangle(x) = \left(\int \frac{d\sigma_2(t)}{x-t} \right) d\sigma_1(x).$$

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Definition (Nikishin system for general r)

A Nikishin system of order r on an interval Δ_r is a system of r measures $(\mu_1, \mu_2, \ldots, \mu_r)$ supported on Δ_r such that $\mu_j = \langle \mu_1, \sigma_j \rangle$ $(2 \leq j \leq r)$, where $(\sigma_2, \ldots, \sigma_r)$ is a Nikishin system of order $r-1$ on an interval Δ_{r-1} and $\Delta_r \cap \Delta_{r-1} = \emptyset$.

Theorem (Fidalgo Prieto and López Lagomasino)

Every Nikishin system is perfect.

Biorthogonality

In most cases the measures $(\mu_1, \ldots, \mu_r)$ are absolutely continuous with respect to one fixed measure μ:

$$d\mu_j(x) = w_j(x)\, d\mu(x), \qquad 1 \leq j \leq r.$$

We then define the **type I function**

$$Q_{\vec{n}}(x) = \sum_{j=1}^{r} A_{\vec{n},j}(x) w_j(x).$$

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Property (biorthogonality)

$$\int P_{\vec{n}}(x) Q_{\vec{m}}(x)\, d\mu(x) = \begin{cases} 0, & \text{if } \vec{m} \leq \vec{n}, \\ 0, & \text{if } |\vec{n}| \leq |\vec{m}| - 2, \\ 1, & \text{if } |\vec{n}| = |\vec{m}| - 1. \end{cases}$$

Nearest neighbor recurrence relations for type II MOPS

$$xP_{\vec{n}}(x) = P_{\vec{n}+\vec{e}_1}(x) + b_{\vec{n},1}P_{\vec{n}}(x) + \sum_{j=1}^{r} a_{\vec{n},j}P_{\vec{n}-\vec{e}_j}(x),$$

$$\vdots$$

$$xP_{\vec{n}}(x) = P_{\vec{n}+\vec{e}_r}(x) + b_{\vec{n},r}P_{\vec{n}}(x) + \sum_{j=1}^{r} a_{\vec{n},j}P_{\vec{n}-\vec{e}_j}(x).$$

$$\vec{e}_j = (0,\ldots,0,\overbrace{1}^{j},0,\ldots,0)$$

Recurrence relations

Nearest neighbor recurrence relations for type I MOPS

$$xQ_{\vec{n}}(x) = Q_{\vec{n}-\vec{e}_1}(x) + b_{\vec{n}-\vec{e}_1,1} Q_{\vec{n}}(x) + \sum_{j=1}^{r} a_{\vec{n},j} Q_{\vec{n}+\vec{e}_j}(x),$$

$$\vdots$$

$$xQ_{\vec{n}}(x) = Q_{\vec{n}-\vec{e}_r}(x) + b_{\vec{n}-\vec{e}_r,r} Q_{\vec{n}}(x) + \sum_{j=1}^{r} a_{\vec{n},j} Q_{\vec{n}+\vec{e}_j}(x).$$

Recurrence relations

Theorem (Van Assche)

The recurrence coefficients $(a_{\vec{n},1},\ldots,a_{\vec{n},r})$ *and* $(b_{\vec{n},1},\ldots,b_{\vec{n},r})$ *satisfy the partial difference equations*

$$\begin{aligned} b_{\vec{n}+\vec{e}_i,j} - b_{\vec{n},j} &= b_{\vec{n}+\vec{e}_j,i} - b_{\vec{n},i} \\ \sum_{k=1}^{r} a_{\vec{n}+\vec{e}_j,k} - \sum_{k=1}^{r} a_{\vec{n}+\vec{e}_i,k} &= \det \begin{pmatrix} b_{\vec{n}+\vec{e}_j,i} & b_{\vec{n},i} \\ b_{\vec{n}+\vec{e}_i,j} & b_{\vec{n},j} \end{pmatrix}, \\ \frac{a_{\vec{n},i}}{a_{\vec{n}+\vec{e}_j,i}} &= \frac{b_{\vec{n}-\vec{e}_i,j} - b_{\vec{n}-\vec{e}_i,i}}{b_{\vec{n},j} - b_{\vec{n},i}} \end{aligned}$$

for all $1 \le i \neq j \le r$.

Recurrence relations

Let $(\vec{n}_k)_{k\geq 0}$ be a path in $\mathbb{N}^r$ starting from $\vec{n}_0 = \vec{0}$, such that $\vec{n}_{k+1} - \vec{n}_k = \vec{e}_i$ for some $1 \leq i \leq r$. Then

$$xP_{\vec{n}_k}(x) = P_{\vec{n}_{k+1}}(x) + \sum_{j=0}^{r} \beta_{\vec{n}_k,j} P_{\vec{n}_{k-j}}(x).$$

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An important case is the **stepline**:

$$\vec{n}_k = (\overbrace{i+1,\ldots,i+1}^{j}, \underbrace{i,\ldots i}_{r-j}) \qquad k = ri+j,\ 0 \leq j \leq r-1.$$

Christoffel-Darboux formula

Theorem (Daems and Kuijlaars)

Let $(\vec{n}_k)_{0 \le k \le N}$ *be a path in* $\mathbb{N}^r$ *starting from* $\vec{n}_0 = \vec{0}$ *and ending in* $\vec{n}_N = \vec{n}$ *(where* $N = |\vec{n}|$*), such that* $\vec{n}_{k+1} - \vec{n}_k = \vec{e}_i$ *for some* $1 \le i \le r$. *Then*

$$(x-y) \sum_{k=0}^{N-1} P_{\vec{n}_k}(x) Q_{\vec{n}_{k+1}}(y) = P_{\vec{n}}(x) Q_{\vec{n}}(y) - \sum_{j=1}^{r} a_{\vec{n},j} P_{\vec{n}-\vec{e}_j}(x) Q_{\vec{n}+\vec{e}_j}(y).$$

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$$(x-y)\sum_{k=0}^{N-1} P_{\vec{n}_k}(x)Q_{\vec{n}_{k+1}}(y) = P_{\vec{n}}(x)Q_{\vec{n}}(y)-\sum_{j=1}^{r} a_{\vec{n},j}P_{\vec{n}-\vec{e}_j}(x)Q_{\vec{n}+\vec{e}_j}(y).$$

The sum depends only on the endpoints of the path in $\mathbb{N}^r$ and not on the path between these points.

Hermite-Padé approximation

Let $(f_1, \ldots, f_r)$ be r Markov functions, i.e.,

$$f_j(z) = \int \frac{d\mu_j(x)}{z - x} = \sum_{k=0}^{\infty} \frac{m_k^{(j)}}{z^{k+1}}.$$

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Definition (Type I Hermite-Padé)

Type I Hermite-Padé approximation is to find r polynomials $(A_{\vec{n},1}, \ldots, A_{\vec{n},r})$, with $\deg A_{\vec{n},j} \leq n_j - 1$, and a polynomial $B_{\vec{n}}$ such that

$$\sum_{j=1}^{r} A_{\vec{n},j}(z) f_j(z) - B_{\vec{n}}(z) = \mathcal{O}\left(\frac{1}{z^{|\vec{n}|}}\right), \qquad z \to \infty.$$

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$$B_{\vec{n}}(z) = \int \sum_{j=1}^{r} \frac{A_{\vec{n},j}(z) - A_{\vec{n},j}(x)}{z-x}\, d\mu_j(x).$$

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$$P_{\vec{n}}(z) f_j(z) - Q_{\vec{n},j}(z) = \mathcal{O}\left(\frac{1}{z^{n_j+1}}\right), \qquad z \to \infty,$$

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$$Q_{\vec{n},j}(z) = \int \frac{P_{\vec{n}}(z) - P_{\vec{n}}(x)}{z-x} \, d\mu_j(x).$$

Multiple Hermite polynomials

The type II multiple Hermite polynomials $H_{\vec{n}}$ satisfy

$$\int_{-\infty}^{\infty} H_{\vec{n}}(x) x^k e^{-x^2+c_j x}\, dx = 0, \qquad 0 \leq k \leq n_j - 1$$

for $1 \leq j \leq r$, with $c_i \neq c_j$ whenever $i \neq j$.

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Rodrigues formula:

$$e^{-x^2} H_{\vec{n}}(x) = \frac{(-1)^{|\vec{n}|}}{2^{|\vec{n}|}} \left(\prod_{j=1}^{r} e^{-c_j x} \frac{d^{n_j}}{dx^{n_j}} e^{c_j x} \right) e^{-x^2}.$$

Explicit expression:

$$H_{\vec{n}}(x) = \frac{(-1)^{|\vec{n}|}}{2^{|\vec{n}|}} \sum_{k_1=0}^{n_1} \cdots \sum_{k_r=0}^{n_r} \binom{n_1}{k_1} \cdots \binom{n_r}{k_r} c_1^{n_1-k_1} \cdots c_r^{n_r-k_r} (-1)^{|\vec{k}|} H_{|\vec{k}|}(x),$$

where H_n are the usual Hermite polynomials.

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Nearest neighbor recurrence relations:

$$xH_{\vec{n}}(x) = H_{\vec{n}+\vec{e}_k}(x) + \frac{c_k}{2} H_{\vec{n}}(x) + \frac{1}{2} \sum_{j=1}^{r} n_j H_{\vec{n}-\vec{e}_j}(x), \qquad 1 \leq k \leq r.$$

Differential properties

Raising operators:

$$\left(e^{-x^2+c_jx}H_{\vec{n}-\vec{e}_j}(x)\right)' = -2e^{-x^2+c_jx}H_{\vec{n}}(x), \qquad 1 \leq j \leq r.$$

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Differential equation:

$$\left(\prod_{j=1}^{r} D_j\right) D H_{\vec{n}}(x) = -2 \left(\sum_{j=1}^{r} n_j \prod_{i \neq j} D_i\right) H_{\vec{n}}(x),$$

where

$$D = \frac{d}{dx}, \qquad D_j = e^{x^2-c_jx} D e^{-x^2+c_jx}$$

Integral representations

Type II multiple Hermite:

$$H_{\vec{n}}(x) = \frac{1}{\sqrt{\pi} i} \int_{-i\infty}^{i\infty} e^{(s-x)^2} \prod_{j=1}^{r} \left(s - \frac{c_j}{2} \right)^{n_j} \, ds.$$

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Type I multiple Hermite:

$$e^{-x^2+c_k x} A_{\vec{n},k}(x) = \frac{1}{\sqrt{\pi} 2\pi i} \oint_{\Gamma_k} e^{-(t-x)^2} \prod_{j=1}^{r} \left(t - \frac{c_j}{2}\right)^{-n_j} \, dt$$

where Γ_k is a closed contour encircling $c_k/2$ once and none of the other $c_j/2$.

Integral representations

Type II multiple Hermite:

$$H_{\vec{n}}(x) = \frac{1}{\sqrt{\pi} i} \int_{-i\infty}^{i\infty} e^{(s-x)^2} \prod_{j=1}^{r} \left(s - \frac{c_j}{2}\right)^{n_j} \, ds.$$

Type I multiple Hermite:

$$e^{-x^2+c_k x} A_{\vec{n},k}(x) = \frac{1}{\sqrt{\pi} 2\pi i} \oint_{\Gamma_k} e^{-(t-x)^2} \prod_{j=1}^{r} \left(t - \frac{c_j}{2}\right)^{-n_j} \, dt$$

where Γ_k is a closed contour encircling $c_k/2$ once and none of the other $c_j/2$.

$$Q_{\vec{n}}(x) = \sum_{k=1}^{r} e^{-x^2+c_k x} A_{\vec{n},k}(x) = \frac{1}{\sqrt{\pi} 2\pi i} \oint_{\Gamma} e^{-(t-x)^2} \prod_{j=1}^{r} \left(t - \frac{c_j}{2}\right)^{-n_j} \, dt$$

where Γ is a closed contour encircling all $c_j/2$.

Random matrices with external source

Let M be a random Hermitian matrix of size $N \times N$, and consider the ensemble with probability distribution

$$\frac{1}{Z_N} \exp\Big(-\mathrm{Tr}(M^2 - AM)\Big)\ dM, \qquad dM = \prod_{i=1}^{N} dM_{i,i} \prod_{1 \le i < j \le N} dM_{i,j}$$

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where A is a fixed Hermitian matrix (the external source).

Property

Suppose A has eigenvalues $c_1, \ldots, c_r$ with multiplicities $n_1, \ldots, n_r$, then

$$\mathbb{E}\Big(\det(M - zI_N)\Big) = (-1)^{|\vec{n}|} H_{\vec{n}}(z).$$

Random matrices with external source

Property

The density of the eigenvalues is given by

$$P_N(\lambda_1, \ldots, \lambda_N) = \frac{1}{N!} \det\Big(K_N(\lambda_i, \lambda_j)\Big)_{i,j=1}^{N},$$

where the kernel is given by

$$K_N(x, y) = e^{-(x^2+y^2)/2} \sum_{k=0}^{N-1} H_{\vec{n}_k}(x) Q_{\vec{n}_{k+1}(y)},$$

with $(\vec{n}_k)_{0 \le k \le N}$ *a path from* $\vec{0}$ *to* $\vec{n}$ *in* $\mathbb{N}^r$ *and*

$$Q_{\vec{n}}(y) = \sum_{j=1}^{r} A_{\vec{n},j}(y) e^{c_j y}.$$

Random matrices with external source

Property

The m-point correlation function

$$R_m(\lambda_1, \dots, \lambda_m) = \frac{N!}{(N-m)!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} P_N(\lambda_1, \dots, \lambda_N)\, d\lambda_{m+1} \dots d\lambda_N$$

is given by

$$R_m(\lambda_1, \dots, \lambda_m) = \det\Big(K_N(\lambda_i, \lambda_j)\Big)_{i,j=1}^{m},$$

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Non-intersecting Brownian motions

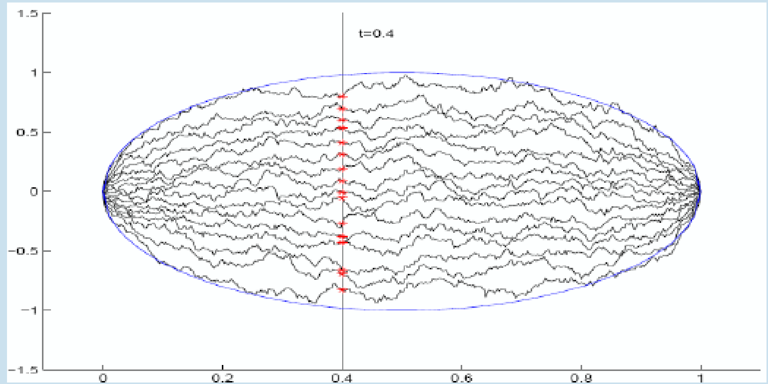

Figure: Non-intersecting Brownian motions

Non-intersecting Brownian motions

Density of the probability that the n non-intersecting paths, leaving $(t = 0)$ at $a_1, \ldots, a_n$ and arriving $(t = 1)$ at $b_1, \ldots, b_n$ are at $x_1, \ldots, x_n$ at time $t \in (0, 1)$ is[1]

$$p_{n,t}(x_1, \ldots, x_n) = \frac{1}{Z_n} \det\Big(P(t, a_j, x_k)\Big)_{j,k=1}^{n} \det\Big(P(1-t, b_j, x_k)\Big)_{j,k=1}^{n}$$

$$P(t, a, x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2t}(x-a)^2}.$$

[1] S. Karlin, J. McGregor: *Coincidence probabilities*, Pacific J. Math. **9** (1959), 1141–1164

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$$P(t,a,x)=\frac{1}{\sqrt{2\pi t}}e^{-\frac{1}{2t}(x-a)^2}.$$

When $a_1,\ldots,a_n\to 0$ and $b_1,\ldots,b_n\to 0$ then

$$p_{n,t}(x_1,\ldots,x_n)=\frac{1}{n!}\det\Big(K_n(x_j,x_k)\Big)_{j,k=1}^n$$

$$K_n(x,y)=e^{-\frac{x^2}{4t}-\frac{y^2}{4(1-t)}}\sum_{k=0}^{n-1}H_k\Big(\frac{x}{\sqrt{2t}}\Big)H_k\Big(\frac{y}{\sqrt{2(1-t)}}\Big)$$

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Non-intersecting Brownian motions

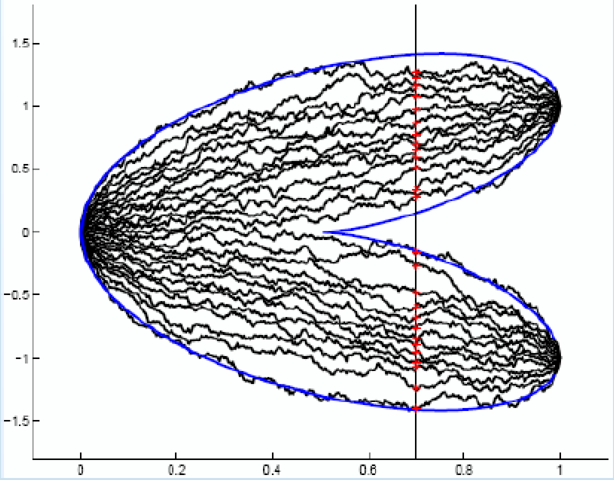

Figure: Non-intersecting Brownian motions (two arriving points)

Non-intersecting Brownian motions

When $a_1, \dots, a_n \to 0$ and $b_1, \dots, b_{n/2} \to -b$, $b_{n/2+1}, \dots, b_n \to b$ then

$$p_{n,t}(x_1, \dots, x_n) = \frac{1}{n!} \det\Big(K_n(x_j, x_k)\Big)_{j,k=1}^{n}$$

$$K_n(x,y) = e^{-\frac{x^2}{4t} - \frac{y^2}{4(1-t)}} \sum_{k=0}^{n-1} H_{\vec{n}_k}(\frac{x}{\sqrt{2t}}) Q_{\vec{n}_{k+1}}(\frac{y}{\sqrt{2(1-t)}})$$

with multiple orthogonal polynomials for the weights

$$e^{-x^2-2bx}, \quad e^{-x^2+2bx}$$

Multiple Laguerre polynomials

The Laguerre weight is

$$w(x) = x^\alpha e^{-x}, \qquad x \in [0,\infty),\ \alpha > -1.$$

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There are two easy ways to obtain multiple Laguerre polynomials:

1. Changing the parameter α to $\alpha_1, \ldots, \alpha_r$
   This gives **multiple Laguerre polynomials of the first kind**
2. Changing the exponential decay at infinity from e^{-x} to $e^{-c_j x}$ with parameters $c_1, \ldots, c_r$
   This gives **multiple Laguerre polynomials of the second kind**

Multiple Laguerre I

Type II multiple Laguerre of the first kind: $L_{\vec{n}}^{\vec{\alpha}}(x)$

$$\int_0^\infty x^k L_{\vec{n}}^{\vec{\alpha}}(x) x^{\alpha_j} e^{-x}\, dx = 0, \qquad 0 \leq k \leq n_j - 1,$$

for $1 \leq j \leq r$.

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Rodrigues formula:

$$(-1)^{|\vec{n}|} e^{-x} L_{\vec{n}}^{\vec{\alpha}}(x) = \prod_{j=1}^{r} \left(x^{-\alpha_j} \frac{d^{n_j}}{dx^{n_j}} x^{n_j+\alpha_j} \right) e^{-x}.$$

Explicit formula:

$$L_{\vec{n}}^{\vec{\alpha}}(x) = \sum_{k_1=0}^{n_1} \cdots \sum_{k_r=0}^{n_r} (-1)^{|\vec{k}|} \frac{n_1!}{(n_1-k_1)!} \cdots \frac{n_r!}{(n_r-k_r)!}$$

$$\times \binom{n_r+\alpha_r}{k_r} \binom{n_r+n_{r-1}+\alpha_{r-1}-k_r}{k_{r-1}} \cdots \binom{|\vec{n}|-|\vec{k}|+k_1+\alpha_1}{k_1} x^{|\vec{n}|-|\vec{k}|}$$

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$$(-1)^{|\vec{n}|} e^{-x} L_{\vec{n}}^{\vec{\alpha}}(x) = \prod_{j=1}^{r} (\alpha_j+1)_{n_j} \, {}_rF_r \left(\begin{matrix} n_1+\alpha_1+1, \ldots, n_r+\alpha_r+1 \\ \alpha_1+1, \ldots, \alpha_r+1 \end{matrix} \middle| -x \right).$$

Multiple Laguerre I: recurrence relation

Recurrence relation:

$$xL_{\vec{n}}(x) = L_{\vec{n}+\vec{e}_k}(x) + b_{\vec{n},k} L_{\vec{n}}(x) + \sum_{j=1}^{r} a_{\vec{n},j} L_{\vec{n}-\vec{e}_j}(x)$$

$$a_{\vec{n},j} = n_j(n_j+\alpha_j) \prod_{i=1, i\neq j}^{r} \frac{n_j+\alpha_j-\alpha_i}{n_j-n_i+\alpha_j-\alpha_i},$$

$$b_{\vec{n},k} = |\vec{n}| + n_k + \alpha_k + 1.$$

Multiple Laguerre I: differential properties

Raising operators:

$$\frac{d}{dx}\left(x^{\alpha_j+1}e^{-x}L_{\vec{n}-\vec{e}_j}^{\vec{\alpha}+\vec{e}_j}(x)\right) = -x^{\alpha_j}e^{-x}L_{\vec{n}}^{\vec{\alpha}}(x), \qquad 1 \leq j \leq r.$$

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Lowering operator:

$$\frac{d}{dx}L_{\vec{n}}^{\vec{\alpha}}(x) = \sum_{j=1}^{r} \frac{\prod_{i=1}^{r}(n_i+\alpha_i-\alpha_j)}{\prod_{i=1,i\neq j}^{r}(\alpha_i-\alpha_j)} L_{\vec{n}-\vec{e}_j}^{\vec{\alpha}+\vec{e}_j}(x).$$

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Differential equation:

$$\left(\prod_{j=1}^{r} D_j\right) DL_{\vec{n}}^{\vec{\alpha}}(x) = -\sum_{j=1}^{r} \frac{\prod_{i=1}^{r}(n_i+\alpha_i-\alpha_j)}{\prod_{i=1,i\neq j}^{r}(\alpha_i-\alpha_j)} \left(\prod_{i\neq j} D_i\right) L_{\vec{n}}^{\vec{\alpha}}(x).$$

$$D = \frac{d}{dx}, \qquad D_j = x^{-\alpha_j}e^{x}Dx^{\alpha_j+1}e^{-x}.$$

Multiple Laguerre II

Type II multiple Laguerre polynomials of the second kind $L_{\vec{n}}^{\alpha,\vec{c}}(x)$

$$\int_0^\infty x^k L_{\vec{n}}^{\alpha,\vec{c}}(x) x^\alpha e^{-c_j x}\, dx = 0, \qquad 0 \leq k \leq n_j - 1,$$

for $1 \leq j \leq r$.

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Rodrigues formula:

$$(-1)^{|\vec{n}|} \prod_{j=1}^r c_j^{n_j}\ x^\alpha L_{\vec{n}}^{\alpha,\vec{c}}(x) = \prod_{j=1}^r \left(e^{c_j x} \frac{d^{n_j}}{dx^{n_j}} e^{-c_j x} \right)\ x^{|\vec{n}|+\alpha}.$$

Multiple Laguerre II

Explicit expression:

$$L_{\vec{n}}^{\alpha,\vec{c}}(x) = \sum_{k_1=0}^{n_1} \cdots \sum_{k_r=0}^{n_r} \binom{n_1}{k_1} \cdots \binom{n_r}{k_r} \binom{|\vec{n}|+\alpha}{|\vec{k}|} (-1)^{|\vec{k}|} \frac{|\vec{k}|!}{c_1^{k_1} \cdots c_r^{k_r}} x^{|\vec{n}|-|\vec{k}|}.$$

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$$a_{\vec{n},j} = \frac{n_j(|\vec{n}|+\alpha)}{c_j^2}, \quad b_{\vec{n},k} = \frac{|\vec{n}|+\alpha+1}{c_k} + \sum_{j=1}^{r} \frac{n_j}{c_j}.$$

Multiple Laguerre II: differential properties

Raising operators:

$$\frac{d}{dx} x^{\alpha+1} e^{-c_j x} L_{\vec{n}-\vec{e}_j}^{\alpha+1,\vec{c}}(x) = -c_j x^{\alpha} e^{-c_j x} L_{\vec{n}}^{\alpha,\vec{c}}(x), \qquad 1 \leq j \leq r.$$

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Differential equation:

$$\left(\prod_{j=1}^{r} D_j\right) x^{\alpha+1} D L^{\alpha,\vec{c}}_{\vec{n}}(x) = -\sum_{j=1}^{r} c_j n_j \left(\prod_{i\neq j} D_i\right) x^{\alpha} L^{\alpha,\vec{c}}_{\vec{n}}(x),$$

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Random matrices: Wishart ensemble

John Wishart (1928) introduced the **Wishart distribution** for $N \times N$ positive definite Hermitian matrices

$$M = XX^*, \qquad X \in \mathbb{C}^{N\times(N+p)}$$

where all the columns of X are independent and have a multivariate Gauss distribution with covariance matrix Σ.

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