Symbolic Computation with Differential Operators

$$y''(x) + (x^2 - 1)y'(x) + y(x) = 0$$

$$\partial^2 + (x^2 - 1)\partial + 1$$

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Differential operators in analysis

Linear ordinary differential equation (ODE)

$$y''(x) + \frac{3}{x}y'(x) + \frac{1}{x^2}y(x) = 0 \qquad (*)$$

$$\partial \quad \text{act as} \quad (\partial f)(x) = f'(x)$$

$(*)$ in terms of ∂

$$\partial^2 y + \frac{3}{x}\partial y + \frac{1}{x^2}y = 0$$

corresponding differential operator

$$L = \partial^2 + \frac{3}{x}\partial + \frac{1}{x^2}$$

(*) in terms of L

$$Ly = 0$$

differential operators as a compact way to write ODEs

view differential operators as algebraic and algorithmic objects

Differential operators with constant coefficients

ODEs with constant coefficients

$$y''(x) - y(x) = 0$$

corresponding differential operator

$$L = \partial^2 - 1$$

view L as a univariate polynomial in ∂ (characteristic equation) and factor it

$$\partial^2 - 1 = (\partial + 1)(\partial - 1)$$

solutions

$$y_1(x) = \exp(-x) \quad \text{and} \quad y_2(x) = \exp(x)$$

Differential operators with constant coefficients =
univariate Polynomial ring

Euclidian ring

Differential rings

Additivity $\frac{d}{dx}(f(x)+g(x)) = f'(x)+g'(x)$

Leibniz rule

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x).$$

Definition

Let R be a commutative ring and $\partial\colon R \to R$ be an additive operation on R such that the Leibniz rule

$$\partial(fg) = \partial(f)g + f\partial(g)$$

holds for any $f, g \in R$.
Then (R, ∂) is called a **differential ring** and ∂ a **derivation** on R.
The **constants** of (R, ∂) are given by

$$C = \{c \in R \mid \partial(c) = 0\}.$$

Constants form a subring and the derivation is linear over the constants

Examples of differential rings

Example

For a commutative coefficient ring C,
polynomials, rational functions, formal power series, and Laurent series

$$C[x], \quad C(x), \quad C[[x]], \quad C((x))$$

with the usual derivation

$$\partial = \frac{d}{dx}$$

are differential rings with constants C.

Example

- exponential polynomials,
- smooth functions, and
- analytic functions

over the real or complex numbers are differential rings.

Differential equations over differential rings

Differential ring (R, ∂) with constants C

Linear differential equations

$$a_n \partial^n y + a_{n-1} \partial^{n-1} y + \cdots + a_0 y = f \qquad (*)$$

with coefficients $a_0, \ldots, a_n \in R$ and right-hand side $f \in R$

solutions y in R or in differential extensions of (R, ∂)

corresponding differential operator

$$L = a_n \partial^n + a_{n-1} \partial^{n-1} + \cdots + a_0$$

$(*)$ in terms of L as

$$Ly = f$$

L is a C-linear operator

Ring of differential operators

Consider the set $R\langle\partial\rangle$ of formal sums in the **indeterminate** ∂ of the form

$$\sum_{i=0}^{n} a_i \partial^i,$$

$n \in \mathbb{N}$ and $a_0, \ldots, a_n \in R$ with usual addition

Multiplication should correspond to composition of operators

Composition of differential operator ∂ and multiplication operator a acts by

$$(\partial \circ a)y = \partial(ay) = \partial(a)y + a\partial(y) = (a\partial + \partial(a))y$$

Definition

Let (R, ∂) be a differential ring. The **ring of linear differential operators** is $R\langle\partial\rangle$ with the noncommutative multiplication defined by the relations

$$\partial \circ a = a \circ \partial + \partial(a)$$

for all $a \in R$.

Algebraic properties

Definition

For a differential operator $L = a_n\partial^n + a_{n-1}\partial^{n-1} + \cdots + a_0 \in R\langle\partial\rangle$ with $a_n \neq 0$ we call n the **order** of L and we call L **monic** if $a_n = 1$.

- R an integral domain, then $R\langle\partial\rangle$ is an integral domain
- R a field, then $R\langle\partial\rangle$ is a left Euclidean domain
- R consists only of constants, then $R\langle\partial\rangle$ is the polynomial ring $R[\partial]$

Example

Let k be a field. Differential operators

- $(k[x])\langle\partial\rangle$ with polynomial coefficients (first **Weyl algebra**)
- $(k(x))\langle\partial\rangle$ with rational function coefficients
- $k\langle\partial\rangle = k[\partial]$ with constant coefficients

Factoring differential operators

Definition

Let (F, ∂) be a differential field and let $L \in F\langle\partial\rangle$. If there are $L_1, L_2 \in F\langle\partial\rangle$ each of order at least 1 such that

$$L = L_1 L_2,$$

then L is called **reducible**, otherwise it is **irreducible**.
If $L = L_1 L_2$, then L_2 is a **right divisor** of L and L is a **left multiple** of L_2.

Example

Consider $(F, \partial) = (\mathbb{C}(x), \frac{d}{dx})$. For any $c \in \mathbb{C}$ the factorization

$$\partial^2 = \left(\partial + \frac{1}{x+c}\right)\left(\partial - \frac{1}{x+c}\right)$$

holds.

Solving factored differential operators

Theorem

Let (F, ∂) be a differential field. Let $L_1, L_2 \in F\langle\partial\rangle$ each with order ≥ 1 and

$$L = L_1 L_2.$$

Let

$$\{\varphi_1, \ldots, \varphi_m\} \quad \textit{and} \quad \{y_1, \ldots, y_n\}$$

be respectively fundamental systems of L_1 and L_2 in F.
If $z_1, \ldots, z_m \in F$ are such that

$$L_2 z_i = \varphi_i$$

for all $i \in \{1, \ldots, m\}$, then

$$\{y_1, \ldots, y_n, z_1, \ldots, z_m\}$$

is a fundamental system of L.

Factorization into first-order operators

$\mathbb{C}(x)\langle\partial\rangle$ with rational function coefficients over the complex numbers

Given a factorization

$$L = (\partial - r_n)(\partial - r_{n-1})\dots(\partial - r_1)$$

with $r_i \in \mathbb{C}(x)$, a fundamental system $y_1, \dots, y_n$ for L is given by

$$y_i = h_1 \int h_2 \int \cdots \int h_i,$$

where $h_j = e^{\int (r_j - r_{j-1})}$ with $r_0 = 0$

Example

$$L = \partial^2 + \frac{3}{x}\partial + \frac{1}{x^2} = \left(\partial + \frac{2}{x}\right)\left(\partial + \frac{1}{x}\right)$$

$$y_1 = e^{-\int \frac{1}{x}} = \frac{1}{x} \qquad y_2 = e^{-\int \frac{1}{x}} \int e^{\int (\frac{1}{x} - \frac{2}{x})} = \frac{1}{x} \int \frac{1}{x} = \frac{\ln(x)}{x}$$

Euclidian division and algorithm

Theorem

Let (F, ∂) *be a differential field. Let* $A, B \in F\langle\partial\rangle$ *be differential operators with* $B \neq 0$*. Then there exist unique* $Q, R \in F\langle\partial\rangle$ *such that*

$$A = QB + R$$

and such that the order of R *is smaller than the order of* B*.*

Example

$$\partial^2 + \frac{3}{x}\partial + \frac{1}{x^2} = \left(\partial + \frac{2}{x}\right)\left(\partial + \frac{1}{x}\right) + 0$$

Theorem

The differential operators $F\langle\partial\rangle$ *over a differential field* F *form a left PID. The greatest common right divisors (GCRD) and least common left multiples (LCLM) can be computed by the extended Euclidean algorithm.*

Systems, PDEs, and generalizations

Systems of linear ODEs

- correspond to matrices over differential operators
- row echelon form (Hermite normal form)
- diagonal form (Jacobson normal form)

PDEs correspond to differential operators in several indeterminates

Example

Second Weyl algebra $k[x, y]\langle\partial_x, \partial_y\rangle$. Multiplication determined by

$$\partial_x x = x\partial_x + 1, \quad \partial_y y = y\partial_y + 1, \quad \partial_x\partial_y = \partial_y\partial x$$

Linear ordinary/partial differential, time-delay, (q)-difference equation are modelled by (iterated) Ore extensions of (skew polynomials over) a ring R with multiplication determined by

$$\partial \circ a = \sigma(a)\partial + \delta(a)$$

Applications of differential operators

Symbolic computation with definite **integrals, sums, and identities** for special functions and combinatorial sequences

(Zeilberger '90, Petkovšek-Wilf-Zeilberger '96, Chyzak-Salvy '98, Koepf '98 '14, Kauers-Koutschan-Zeilberger '09, Kauers-Paule '11)

Algebraic systems theory for linear systems of ordinary/partial differential, time-delay, and difference equations

Matrices/modules of/over polynomial and Ore algebras

(Oberst '90, Pommaret-Quadrat '03, Chyzak-Quadrat-Robertz '05, Gómez-Torrecillas '14, Robertz '15, Seiler-Zerz '15)

Euclidian algorithm and (non)commutative **Gröbner bases**

(Ore '33, Buchberger '65, Kandri-Rody-Weispfenning '90, Kredel '93, Bronstein-Petkovšek '96, Chyzak-Salvy '98, Chyzak '98, Bueso-Gómez-Torrecillas-Verschoren '03, Levandovskyy '06, Koutschan '10)

Dixmier conjecture: Any **endomorphism** of the Weyl algebra is **invertible**.

(Dixmier '68)

Equivalent to Jacobian conjecture

(Tsuchimoto '05, Belov-Kanel-Kontsevich '07, Adjamagbo-van den Essen '07)

Tutorial exercise 1

Use the command `desolve` to solve

$$y'(x) + y \quad \text{and} \quad y'(x) + y = e^x$$

in Sage

Introduce the function $y(x)$ in Sage by the command

```
y = function('y')(x)
```

Tutorial exercise 2

Let (R, ∂) be a commutative differential ring, that is, $\partial\colon R \to R$ is additive and satisfies the Leibniz rule

$$\partial(fg) = \partial(f)g + f\partial(g)$$

for all $f, g \in R$. Verify that the constants of R given by

$$C = \{c \in R \mid \partial(c) = 0\}$$

form a subring of R and that the derivation ∂ is linear over the constants C.

Solution for exercise 2

Let $a, b \in C$, then $a + b \in C$, since

$$\partial(a + b) = \partial(a) + \partial(b) = 0 + 0 = 0.$$

Since

$$\partial(ab) = \partial(a)b + a\partial(b) = 0 \cdot b + a \cdot 0 = 0,$$

also $ab \in C$.

$$\partial(0) = \partial(0 + 0) = \partial(0) + \partial(0),$$

hence $\partial(0) = 0$ and $0 \in C$.

Solution for exercise 2

Moreover,

$$0 = \partial(a - a) = \partial(a) + \partial(-a) = \partial(-a)$$

hence also $-a \in C$.

$$\partial(1) = \partial(1\cdot 1) = \partial(1)\cdot 1 + 1 \cdot \partial(1) = \partial(1) + \partial(1)$$

which implies $\partial(1) = 0$ and hence $1 \in C$.

∂ is additive and C-linearity of ∂ follows for $c \in C$ and $a \in R$ from

$$\partial(ca) = \partial(c)a + c\partial(a) = 0 \cdot a + c\partial(a) = c\partial(a).$$

Tutorial exercise 3

```
from ore_algebra import *
R.<x> = PolynomialRing(QQ)
K=R.fraction_field()
B.<Dx>=OreAlgebra(K)
```

Multiply the differential operator `Dx` with the multiplication operator `x`.

Define the differential operator

$$L = Dx^2 + ((-x+1)/x) * Dx + (-x-1)/x^2$$

and try to solve $L(y) = 0$ with the command `desolve`

Tutorial exercise 4

Let (R, ∂) be a commutative differential ring and $y \in R$ invertible.
Find $r \in R$ such that y solves the differential equation defined by

$$B = \partial - r \in R\langle\partial\rangle$$

that is, such that $B(y) = 0$.

Solution for exercise 4

Let

$$r = \partial(y)/y$$

Then

$$B = \partial - \partial(y)/y$$

satisfies $B(y) = 0$

For finding first-order factors with rational coefficients in $k(x)$, we need to find **hyperexponential** solutions satisfying

$$y'(x)/y(x) \in k(x)$$

Rational solutions are a particular case of hyperexponential solutions