Solving Polynomial Systems Using Linear Algebra - Tutorial 1

CAA Workshop 7-13 Oct. 2017 Douala

To solve these exercises, we will use ApCoCoA.

1 Endomorphisms

Exercise 1 Minimal Polynomials

Let $K = \mathbb{Q}$ and let φ be the endomorphism of $V = K^4$ defined by the matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

with respect to the canonical basis.

(a) Compute the minimal polynomial of φ via reduction.

(b) Consider the endomorphism $\psi = \varphi^2 - \varphi$ of V. Compute the minimal polynomial of ψ.

(c) What is the minimal polynomial of $\phi = 2\varphi - 3id_V$?

Exercise 2 Eigenfactors, Eigenvalues, Eigenspaces

In the setting of Exercise **1**, perform the following tasks.

(a) Factor the minimal polynomial of ϕ.

(b) Deduce the eigenvalues, the eigenfactors, and the eigenspaces of ϕ.

(c) What is the kernel of the ideal $\langle \phi^2 - \phi \rangle$ in $K[\phi]$?

Exercise 3 The Generalized Eigenspace Decomposition

Let $K = \mathbb{Q}$ and let φ be the endomorphism of $V = K^8$ represented by the matrix

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

with respect to the canonical basis of V.

(a) Give a K-basis of $\mathrm{Ker}(\varphi)$.

(b) What are $\mathrm{Ker}(\varphi^2)$ and $\mathrm{Ker}(\varphi^3)$?

(c) Deduce a K-basis of $\mathrm{BigKer}(\varphi)$.

(d) Write an ApCoCoA procedure called **BigKer** which computes $\mathrm{BigKer}(\varphi, p(z))$ for a given endomorphism φ and an eigenfactor $p(z)$ of φ.

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2 Commuting Families

Exercise 4 Kernels of Ideals

Let $K = \mathbb{Q}$, let $V = K^4$ and let φ_1, φ_2 be the endomorphisms of V represented by the matrices

$$A_1 = \begin{pmatrix} 0 & \frac{5}{2} & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{5}{2} & 0 & 0 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

respectively.

(a) Show that $\mathcal{F} = K[\varphi_1, \varphi_2]$ is a commuting family.

(b) Let I be the ideal of $\mathcal{F}$ generated by $\{\varphi_1^2 - 5id_V, \varphi_2 - id_V\}$.

 (1) Give the matrices which represent the two generators of I.

 (2) Deduce that I is a proper ideal in $\mathcal{F}$ and compute its kernel.

Exercise 5 Computing $\mathfrak{m}$-Eigenfactors

Let us consider the endomorphism φ given by the matrix A of Exercise **1**.

(a) How many maximal ideals do we have in the family $\mathcal{F} = K[\varphi]$? List them.

(b) Consider the endomorphism $\psi = \varphi^2 - \varphi$ of K^4.

 (1) What is the K-dimension of $K[\psi]$?

 (2) Compute the eigenfactors $p_{\mathfrak{m}_1,\varphi}(z)$, $p_{\mathfrak{m}_2,\varphi}(z)$ of φ and the eigenfactors $p_{\mathfrak{m}_1,\psi}(z)$, $p_{\mathfrak{m}_2,\psi}(z)$ of ψ.

 (3) Compute the eigenspaces $\mathrm{Eig}(\varphi, p_{\mathfrak{m}_2,\varphi}(z))$ and $\mathrm{Eig}(\psi, p_{\mathfrak{m}_2,\psi}(z))$ and interpret the result.

Exercise 6 Computation of Linear Maximal Ideals

Let $K = \mathbb{Q}$, let $V = K^4$, and let φ_1 and φ_2 be the endomorphisms of V defined by the matrices

$$A_1 = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad \text{and} \qquad A_2 = \begin{pmatrix} 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 8 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

respectively.

(a) Show that $\mathcal{F} = K[\varphi_1, \varphi_2]$ is a commuting family.

(b) What are the minimal polynomials of φ_1 and φ_2?

(c) Deduce the eigenfactors of φ_1, φ_2, and deduce the linearity of all maximal ideals of the family $\mathcal{F} = K[\varphi_1, \varphi_2]$.