Solving in Computer Algebra

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  - Solving polynomial equations
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  - Solving modular equations: Chinese remainder theorem, discrete logarithms

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  - Solving modular equations: Chinese remainder theorem, discrete logarithms
  - Solving differential equations

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  - Solving linear systems
  - Solving polynomial equations
  - Solving polynomial systems
  - Solving modular equations: Chinese remainder theorem, discrete logarithms
  - Solving differential equations
  - Solving recurrence equations

Solving linear systems

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- We saw already last week that one can also work with matrices and compute determinants etc.
- Martin Kreuzer will give more details about modern linear algebra techniques in his talk.

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- In algebra it is proved that, in the generic case, the zeros of a polynomial of degree larger than 4 cannot be given by a formula.
- Nevertheless in special cases, polynomial zeros can be computed. For this purpose, polynomial factorization (over $\mathbb{Q}$) is used. This method is accessible via the factor and solve commands.

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Resultants

- Resultants can be used to eliminate variables in polynomial systems.
- Using this method, we solve the system

$$\begin{aligned} x + y + z &= 6 \\ x^2 + y^2 + z^2 &= 14 \\ x^4 + y^4 + z^4 &= 98\,. \end{aligned}$$

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- Several equations of the type

$$\begin{aligned} x &\equiv a_1 \pmod{p_1} \\ x &\equiv a_2 \pmod{p_2} \\ &\vdots \\ x &\equiv a_n \pmod{p_n}, \end{aligned}$$

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- For the solution the extended Euclidean algorithm is used.
- As an example, we solve the system

$$\begin{aligned} x &\equiv 0 \pmod{2} \\ x &\equiv 0 \pmod{3} \\ x &\equiv 1 \pmod{7}\,, \end{aligned}$$

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- Why is it so difficult to compute discrete logarithms?
- To visualize the main reason for this difficulty, we plot the modular exponential function.

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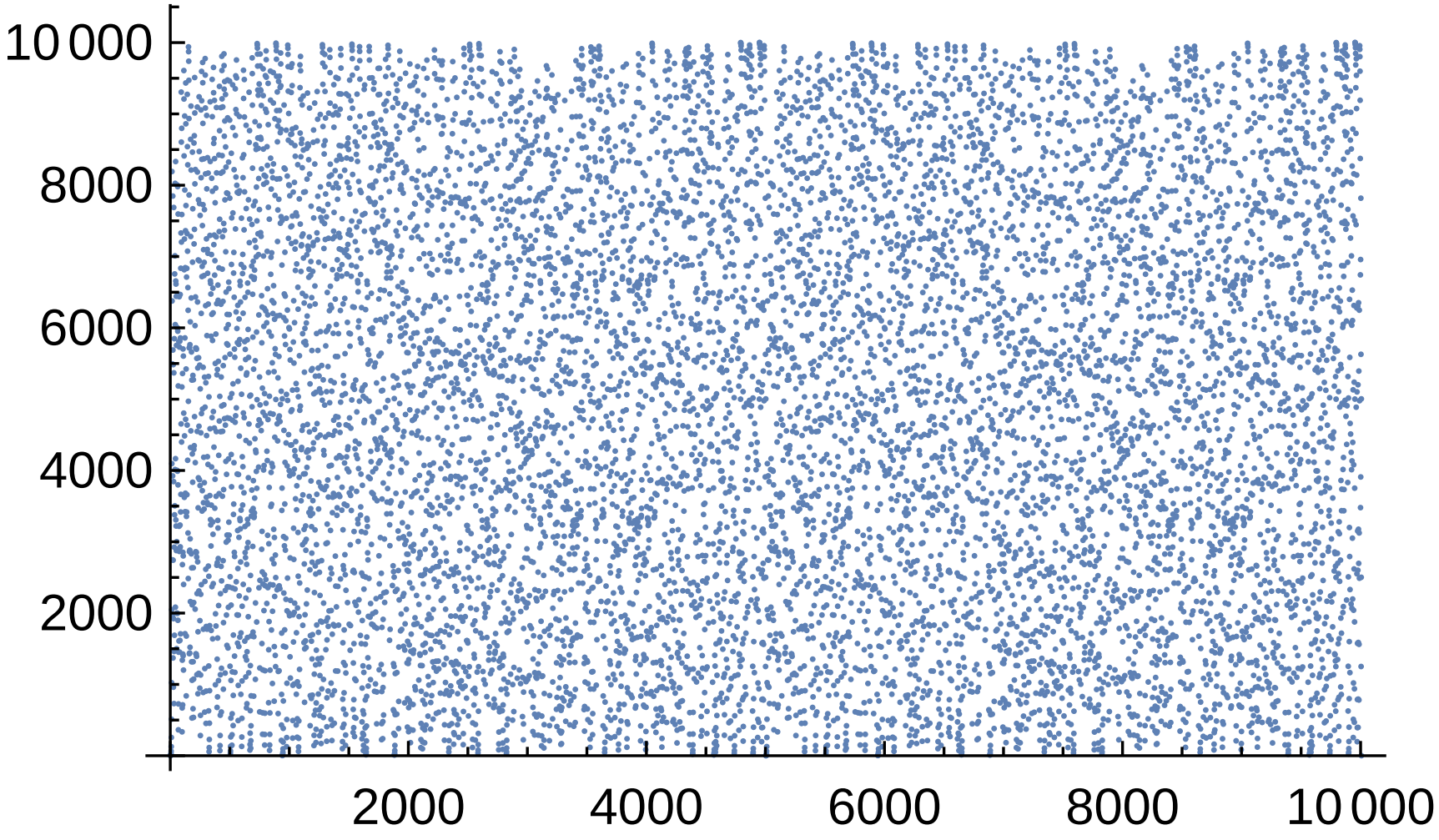

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Recurrence Equations

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- More details will be given in my talk next Friday.

Many thanks for your interest!