AIMS-Volkswagen Stiftung Workshop on Introduction to Orthogonal Polynomials and Applications

5 - 12 October 2018

ABSTRACTS

Editors:

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AIMS-VOLKSWAGEN STIFTUNG WORKSHOP ON INTRODUCTION ORTHOGONAL POLYNOMIALS AND APPLICATIONS
Limbe/Douala, Cameroon, October 5-12, 2018
Preliminary Program of the Second Workshop

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Preliminary Trainings

Preliminary Training 1: *Introduction to Orthogonal Polynomials: Definition and basic properties*

Friday 5 October, 08:30-09:30
Speaker: *Mama Foupouagnigni*

In this introductory talk, we first define the notion of orthogonal polynomials, then provide with illustrations and proof some basic properties such as: the uniqueness of a family of orthogonal polynomials with respect to a weight (up to a multiplicative factor), the matrix representation, the three-term recurrence relation, the Christoffel-Darboux formula and some of its consequences. Finally we discuss and solve, as part of a short tutorial, some assignments.

Preliminary Training 3: *Classical continuous orthogonal polynomials (Part I)*

Friday 5 October, 11:00-12:00
Speaker: *Salifou Mboutngam*

Classical orthogonal polynomials (Hermite, Laguerre, Jacobi and Bessel) constitute the most important families of orthogonal polynomials. They appear in mathematical physics when Sturn-Liouville problems for hypergeometric differential equation are studied. These families of orthogonal polynomials have specific properties. The aim of this training is to:

1. give the definition of classical continuous orthogonal polynomials;
2. prove the orthogonality of the sequence of the derivatives;
3. prove that all terms of the classical orthogonal polynomials sequence satisfy a second order homogeneous linear differential equation;
4. give the Rodrigues formula.

Preliminary Training 5: *Classical continuous orthogonal polynomials (Part II)*

Friday 5 October, 14:30-15:30
Speaker: *Maurice Kenfack*

Let $\{p_n(x)\}_{n=0}^{\infty}$ be a family of classical orthogonal polynomials. The function

$$G(x,t) = \sum_{n=0}^{+\infty} c_n p_n(x) t^n, \; c_n \in \mathbb{R} \tag{1}$$

is called generating function of the set $\{p_n(x)\}_{n=0}^{\infty}$. We show how to obtain generating functions for classical orthogonal polynomials and derive their differential difference equation as well as their hypergeometric representation. We also introduce generalized hypergeometric functions and establish their basic properties.

Preliminary Training 7: *Inversion, multiplication and connection formulae for classical continuous orthogonal polynomials*

Friday 5 October, 17:00-18:00
Speaker: *Daniel Duviol Tcheutia*

Our main objective is to establish the so-called *connection formula*,

$$p_n(x) = \sum_{k=0}^{n} C_k(n) y_k(x), \tag{2}$$

which for $p_n(x) = x^n$ is known as the *inversion formula*

$$x^n = \sum_{k=0}^{n} I_k(n) y_k(x),$$

for the family $y_k(x)$, where $\{p_n(x)\}_{n\in\mathbb{N}_0}$ and $\{y_n(x)\}_{n\in\mathbb{N}_0}$ are two polynomial systems. If we substitute x by ax in the left hand side of (2) and y_k by p_k, we get the *multiplication formula*

$$p_n(ax) = \sum_{k=0}^{n} D_k(n,a) p_k(x).$$

The coefficients $C_k(n)$, $I_k(n)$ and $D_k(n,a)$ exist and are unique since $\deg p_n = n$, $\deg y_k = k$ and the polynomials $\{p_k(x),\ k = 0,1,\ldots,n\}$ or $\{y_k(x),\ k = 0,1,\ldots,n\}$ are linearly independent. In this session, we show how to use generating functions or the structure relations to compute the coefficients $C_k(n)$, $I_k(n)$ and $D_k(n,a)$.

Plenary Training 9: *Properties and applications of the zeros of classical continuous orthogonal polynomials*

Saturday 6 October, 09:30-10:30
Speaker: *Daniel Duviol Tcheutia*

Suppose $\{P_n\}_{n=0}^{\infty}$ is a sequence of polynomials, orthogonal with respect to the weight function $w(x)$ on the interval $[a,b]$. In this lecture, we will prove some basic properties of the zeros of orthogonal polynomials: the location of the zeros, interlacing of zeros of polynomials P_n and P_{n-1}, as well as Stieltjes interlacing of zeros of P_n and $P_{n-m}, m \geq 2$. We will also discuss the main ingredients of the Gauss quadrature formulas, where the zeros of orthogonal polynomials are of decisive importance in approximating integrals.

Preliminary Training 11: *Classical orthogonal polynomials of a discrete and a q-discrete variable*

Saturday 6 October, 11:30-13:00
Speaker: *Patrick Njionou*

The classical orthogonal polynomials of discrete and q-discrete orthogonal polynomials are introduced from their difference and q-difference equations. Some structure formulas are proved for the Charlier and the Al-Salam Carlitz polynomials from their generating functions.

Keywords. Orthogonal polynomials, generating function, inversion formula, connection formula, addition formula, multiplication formula.

Preliminary Training 12: *Computer algebra, power series and summation*

Saturday 6 October, 14:30-15:30
Speaker: *Wolfram Koepf*

Computer algebra systems can do many computations that are relevant for orthogonal polynomials and their representations. In this preliminary training we will introduce some of those important algorithms: The automatic computation of differential equations and formal power series, hypergeometric representations, and the algorithms by Fasenmyer, Gosper, Zeilberger and Petkovsek/van Hoeij.

Preliminary Training 14: *On the solutions of holonomic third-order linear irreducible differential equations in terms of hypergeometric functions*

Saturday 6 October, 17:00-18:00
Speaker: *Merlin Mouafo*

We present here an algorithm that combines change of variables, exp-product and gauge transformation to represent solutions of a given irreducible third-order linear differential operator L, with rational function coefficients and without Liouvillian solutions, in terms of functions $S \in \{{}_1F_1^2,\ {}_0F_2,\ {}_1F_2,\ {}_2F_2\}$ where ${}_pF_q$ with $p \in \{0,1,2\}$, $q \in \{1,2\}$, is the generalized hypergeometric function. That means we find rational functions r, r_0, r_1, r_2, f such that the solution of L will be of the form

$$\exp\left(\int r\,dx\right)\left(r_0 S(f(x)) + r_1 (S(f(x)))' + r_2 (S(f(x)))''\right).$$

An implementation of this algorithm in Maple is available.

Plenary Talks

Plenary Talk 1: *Orthogonal polynomials and random matrices*

Monday 8 October, 08:30-09:30
Speaker: *Walter Van Assche*

Orthogonal polynomials are a very useful tool to investigate eigenvalues or singular values of certain random matrices. We explain a number of random matrix ensembles, such as the Gaussian unitary ensemble (hermitian matrices), the Wishart ensemble (positive definite matrices) and truncations of matrices from Haar measure on unitary matrices. The eigenvalues or singular values of these random matrices behave asymptotically like the zeros of Hermite polynomials, Laguerre polynomials and Jacobi polynomials. We show how one can find the expected value of the characteristic polynomial of such matrices and how one can find the probability density of the eigenvalues in terms of orthogonal polynomials.

Plenary Talk 2: *Review of the one variable theory for polynomials and matrix polynomials orthogonal on the unit circle*

Monday 8 October, 09:30-10:30
Speaker: *Jeff Geronimo*

The main objective of my lectures will be to present some recent results on the theory of bivariate orthogonal polynomials on the bicircle and Fejer-Riesz factorization. In the first lecture I will review the theory of polynomials orthogonal on the unit circle including the derivation of the recurrence formulas satisfied by these polynomials, the stability and spectral matching properties of the reverse polynomials and a derivation of the Christoffel-Darboux formula. I will prove Verblunsky's theorem for these polynomials which implies that there is a one-to-one correspondence between infinite sequences of recurrence coefficients and orthogonality measures and then discuss the Fejer-Riesz lemma which states that any positive trigonometric polynomial can be factored as a magnitude squared of an algebraic polynomial of the same degree. I will also discuss how these results carry over to matrix orthogonal polynomials on the unit circle.

Plenary Talk 3: ***Orthogonality properties of q-special functions (Part I)***
Monday 8 October, 11:00-12:00
Speaker: *Sergei Suslov*

We analyze various orthogonality relations for q-special functions on the basis of q-Sturm-Liouville theory and its extensions. In addition to classical orthogonal polynomials of a discrete variable, our examples will include the Askey-Wilson polynomials, basic Fourier series and other q-orthogonal special functions.

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- S.K. Suslov, An Introduction to Basic Fourier Series, Springer, 2003; `https://www.springer.com/us/book/9781402012211`

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Plenary Talk 4: ***Exceptional orthogonal polynomials***
Monday 8 October, 12:00-13:00
Speaker: *David Gómez-Ullate Oteiza*

Exceptional orthogonal polynomials are orthogonal polynomial systems that satisfy a Sturm-Liouville problem. Until recently it was believed that the only such systems were the classical families of Hermite, Laguerre and Jacobi, but relaxing the condition that a polynomial of every degree is present in the sequence leads to a wider class, that we name *exceptional*. This introductory lecture will review the main results on the theory of exceptional polynomials, obtained by many authors during the past ten years. We will emphasize the similarities and differences with respect to their classical counterparts, while we review results on their construction, orthogonality weight, zeros and asymptotics, recurrence relations, etc. Some key results on their classification will also be given, although the proof of that result will probably exceed the scope of these lectures. The key concepts for the construction are Darboux transformations, factorization of second order differential operators, Wronskian determinants, etc. The details of some calculations will be proposed as exercises, some of which will be solved during the tutorial session.

Plenary Talk 5: ***The factorization method for discrete orthogonal polynomials of hypergeometric type***

Monday 8 October, 14:30-15:30

Speaker: *Gaspard Bangerezako*

We show how to use special types of factorization methods to generate or modify the most important classes of orthogonal polynomials in the Askey and q-Askey scheme: Difference and q-difference hypergeometric polynomials, including the Askey-Wilson polynomials. These polynomials are considered as special cases of solutions of hypergeometric difference and q-difference equations.

Keywords. Hypergeometric difference and q-difference equations, difference and q-difference orthogonal polynomials, factorization methods.

Plenary Talk 6: $(\mathscr{R}, p, q)$***-Rogers-Szegő and Hermite polynomials, and induced deformed quantum algebras***
Monday 8 October, 15:30-16:30
Speaker: *Mahouton Norbert Hounkonnou*

Deformed quantum algebras, namely the $q-$ deformed algebras and their extensions to $(p,q)-$deformed algebras, continue to attract much attention. One of the main reasons is that these topics represent a meeting point of nowadays fast developing areas in mathematics and physics like the theory of quantum orthogonal polynomials and special functions, quantum groups, integrable systems, quantum and conformal field theories and statistics.
This talk aims at characterizing the $(\mathscr{R},p,q)-$Rogers-Szegő polynomials, and the $(\mathscr{R},p,q)-$deformed difference equation giving rise to raising and lowering operators. These polynomials induce some realizations of generalized deformed quantum algebras, (called $(\mathscr{R},p,q)-$deformed quantum algebras), which are also explicitly constructed. The study of continuous $(\mathscr{R},p,q)-$Hermite polynomials is also performed. Known particular cases are recovered.

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Contributed Talk 7: *On orthogonal polynomials with respect to normal operators*

Monday 8 October, 17:00-17:30
Speaker: *Benard Nyaare*

Many important relations in physical sciences are represented by normal operators satisfying various commutation relations. Such commutation relations play key roles in such areas as quantum mechanics, wavelet analysis, spectral theory, representation theory, and many others. In this talk, we present the p-system which is a new result, and establish some connections between the three systems in terms of normal operators J, R and Q. Details of the derivation of the p-system are given. This system of orthogonal polynomials is obtained by applying the Gram-Schmidt procedure to the sequence x_n with respect to the L^2 inner product. It turns out that this system has a simple recurrence relation so that its exponential generating function is easily computed. Using this, the orthogonality of the system is proved. Investigated also are boundedness properties of the normal operators $B = R^{-1}$ and $S = JR^{-1}$ in Hilbert spaces related to the three systems. Orthogonal polynomials used to prove boundedness in the weighted spaces and Fourier analysis are also used to prove boundedness in the translation invariant case. It is proved in both cases that these two operators are bounded on the L^2-spaces, and estimates of the norms are obtained.

Contributed Talk 8: *Three systems of orthogonal polynomials and L2-boundedness of two associated operators*

Monday 8 October, 17:30-18:00
Speaker: *John Musonda*

In this talk, we describe three systems of orthogonal polynomials belonging to the class of Meixner-Pollaczek polynomials, and establish some useful connections between them in terms of three basic operators that are related to them. Furthermore, we investigate boundedness properties of two other operators in the Hilbert spaces related to the orthogonal polynomials. Orthogonal polynomials are used to prove boundedness in the weighted spaces and Fourier analysis is used to prove boundedness in the translation invariant case. It is proved in both cases that the two operators are bounded on L^2-spaces, and estimates of the norms are obtained.

Plenary Talk 9: *Two variable orthogonal polynomials on the bicircle*

Tuesday 9 October, 08:30-09:30
Speaker: *Jeff Geronimo*

The main objective of my lectures will be to present some recent results on the theory of bivariate orthogonal polynomials on the bicircle and Fejer-Riesz factorization. In the second lecture I will discuss bivariate orthogonal polynomials on the bicircle. Using particular orderings (the lexicographical and reverse lexicographical orderings) I will derive recurrence formulas associated with these polynomials and prove a Verblunsky Theorem. A Fejer-Riesz Theorem will be proved showing that a positive bivariate trigonometric polynomial satisfying some side constraints can be factored as the magnitude square of a bivariate algebraic polynomial. This side constraint remarkably can be expressed easily in terms of a coefficient in one of the recurrence formulas satisfied by the polynomials. Time permitting I will discuss the Fourier coefficients of the inverse of the above trigonometric polynomial.
In the tutorial session I will complete any unfinished proofs from the first two lectures and discuss the examples assigned earlier.

Plenary Talk 10: ***Multiple orthogonal polynomials***

Tuesday 9 October, 09:30-10:30
Speaker: *Walter Van Assche*

Multiple orthogonal polynomials are an extension of orthogonal polynomials and satisfy orthogonality relations with respect to several measures (instead of one). Let $\mu_1, \ldots, \mu_r$ be positive measures on the real line and $\vec{n} = (n_1, n_2, \ldots, n_r) \in \mathbb{N}^r$ a multi-index, then the type II multiple orthogonal polynomial $P_{\vec{n}}$ is a monic polynomial of degree $|\vec{n}| = n_1 + n_2 + \cdots + n_r$ for which

$$\int P_{\vec{n}}(x) x^k \, d\mu_j(x) = 0, \qquad 0 \le k \le n_j - 1,$$

for every j for which $1 \le j \le r$. There is also a vector of type I multiple orthogonal polynomials $(A_{\vec{n},1}, \ldots, A_{\vec{n},r})$, where $A_{\vec{n},j}$ has degree $n_j - 1$ and

$$\sum_{j=1}^{r} \int x^k A_{\vec{n},j}(x) \, d\mu_j(x) = 0, \qquad 0 \le k \le |\vec{n}| - 2.$$

We will give a number of examples and some applications where these multiple orthogonal polynomials appear: number theory, rational approximation, random matrices, non-intersecting Brownian motions, etc.

Plenary Talk 11: ***Orthogonality properties of q-special functions (Part II)***

Tuesday 9 October, 11:00-12:00
Speaker: *Sergei Suslov*

We analyze various orthogonality relations for q-special functions on the basis of q-Sturm-Liouville theory and its extensions. In addition to classical orthogonal polynomials of a discrete variable, our examples will include the Askey-Wilson polynomials, basic Fourier series and other q-orthogonal special functions.

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Plenary Talk 12: *Semiclassical orthogonal polynomials*

Tuesday 9 October, 12:00-13:00
Speaker: *Francisco Marcellán*

In this lecture we will present an overview about semiclassical linear functionals with respect to a lowering operator.
A short introduction containing the basic background on orthogonal polynomials following [1] and [2] will yield an updated presentation of the theory on semiclassical linear functionals according to [3].

1. Linear functionals. Moment sequences. Stieltjes functions. Jacobi matrices.
2. Spectral transformations of linear functionals. Darboux transformations
3. Semiclassical linear functionals. Characterizations.
4. Semiclassical linear functionals and linear spectral transformations.
5. A constructive approach to semiclassical linear functionals of classes 1 and 2. Some applications.

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Contributed Talk 15: *Summation formula for generalized discrete q-Hermite II polynomials*

Tuesday 9 October, 17:00-17:30
Speaker: *Sama Arjika*

In this talk, we provide a family of generalized discrete q-Hermite II polynomials denoted by $\tilde{h}_{n,\alpha}(x,y|q)$. Explicit relations connecting them with the q-Laguerre and Stieltjes-Wigert polynomials are obtained. A summation formula is derived by using different analytical means on their generating functions.

Keywords. Basic orthogonal polynomials, Hermite polynomials, discrete q-Hermite II polynomials, connection formulae.

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Contributed Talk 16: *On certain properties of a perturbed Freud-type weight*

Tuesday 9 October, 17:30-18:00
Speaker: *Abey Kelil*

In this paper we study the recurrence coefficients of orthogonal polynomials associated with certain perturbed Freud-type weights. Using the ladder operators and associated compatibility conditions, we derive certain nonlinear difference equations satisfied by the recurrence coefficients of these orthogonal polynomials. These ladder equations extend known results for orthogonal polynomials and they can be used to derive second-order differential equations satisfied by these orthogonal polynomials. Combining these equations with the three-term recurrence relation yields structural relation for such orthogonal polynomials.

Plenary Talk 17: *Maya diagrams and rational solutions to Painlevé equations*

Wednesday 10 October, 08:30-09:30
Speaker: *David Gómez-Ullate Oteiza*

In this lecture we will show how rational solutions to Painlevé equations PIV, PV, and their higher order A_n-Painlevé generalizations can be built in a very simple and direct manner using the concept of cyclic Maya diagrams. Maya diagrams are horizontal diagrams of filled and empty boxes that visually encode the main operations needed for this construction. To every such diagram we can associate a potential in a certain class which is closed under Darboux transformations. To every $(n+1)$-cyclic chain of potentials and Darboux transformations there corresponds a solution to the A_n-Painlevé system, and we will show how to study and classify all such cycles by means of simple operations on the Maya digrams. The result is a convenient way to encode explicitly all such rational solutions, given as log derivatives of Wronskian determinants whose entries are Hermite and Laguerre polynomials. The resulting polynomial families include all known solutions like the generalized Hermite, Okamoto and Umemura polynomials, and they extend to many new classes. Despite the seemingly complex content, the lecture will be surprisingly easy to follow. Detailed calculations will be given as exercises and solved during the tutorial session.

Plenary Talk 18: *Orthogonal polynomials in Sobolev spaces*

Wednesday 10 October, 09:30-10:30
Speaker: *Francisco Marcellán*

In this lecture we will present a nice application of these functionals in the framework of the orthogonality with respect to Sobolev inner products associated with them.
Concerning the orthogonality with respect to a Sobolev inner product we will focus the attention on the univariate case. We will overview some analytic properties of the corresponding orthogonal polynomials, focussing the attention on the distribution of their asymptotic behavior. The case of coherent pairs of measures (see [2]) will be studied with a particular emphasis. As an application of Sobolev orthogonality, spectral methods for boundary value problems for ordinary differential equations will be analyzed.

1. Orthogonality with respect to Sobolev inner products.
2. Sobolev-type orthogonal polynomials. Analytic properties.
3. Coherent pairs of measures and Sobolev orthogonal polynomials.
4. Asymptotics of Sobolev orthogonal polynomials.
5. Fourier expansions in terms of Sobolev orthogonal applications. Spectral methods for boundary value problems in ordinary differential equations and Sobolev orthogonal polynomials.

References

[1] F. Marcellán, Yuan Xu, *On Sobolev orthogonal polynomials*. Expo. Math. **33** (2015), no. 3, 308–352.
[2] H. G. Meijer, *Determination of all coherent pairs*. J. Approx. Theory **89** (1997), no. 3, 321–343.

Plenary Talk 19: *A collection of classical three-fold symmetric 2-orthogonal polynomials*

Wednesday 10 October, 11:00-12:00
Speaker: *Ana Loureiro*

The three-fold symmetry of a polynomial sequence means that $P_n(\omega^k x) = \omega^{nk} P_n(x)$ for all $n \in \mathbb{N}$ with $\omega = e^{\frac{2i\pi}{3}}$ and $k = 0, 1$ and 2. The 2-orthogonal polynomials $\{P_n(x)\}_{n\in\mathbb{N}}$ with respect to a vector of measures (μ_0, μ_1) are multiple orthogonal polynomials of type II $\{\widehat{P}_{\vec{n}}(x)\}_{\vec{n}\in\mathbb{N}^2}$ whose index lies on the step line. So, we can assume $P_{2n}(x) = \widehat{P}_{n,n}(x)$ and $P_{2n+1}(x) = \widehat{P}_{n,n+1}(x)$, for all $n \in \mathbb{N}$. Three-fold symmetric 2-orthogonal polynomials satisfy the third-order recurrence relation

$$P_{n+1}(x) = xP_n(x) - \gamma_{n-1} P_{n-2}(x)$$

with initial conditions $P_k(x) = x^k$ for $k = 0, 1, 2$. The focus of this talk is on the characterisation of all the three-fold symmetric 2-orthogonal polynomials such that the sequence of its derivatives is again 2-orthogonal. This is basically an extension of Hahn's property into the context of 2-orthogonality. Recall that within the (standard) orthogonality context, only the classical polynomials of Hermite, Laguerre, Bessel and Jacobi share this property. So, in a way, we will be discussing an extension of the classical polynomials in the 2-orthogonality context. Among the properties under analysis are the location and asymptotic behaviour of the zeros alongside with the orthogonality measures supported on a three-star of the complex plane. We will show that there are essentially three distinct families of three-fold symmetric 2-orthogonal Hahn-classical polynomials, whose weight functions are described via Airy function, confluent hypergeometric functions and hypergeometric functions, respectively. This is joint work with Walter Van Assche.

Plenary Talk 20: *Zeros of orthogonal polynomials*

Wednesday 10 October, 12:00-13:00
Speaker: *Kerstin Jordaan*

In this lecture we will discuss properties of zeros of orthogonal polynomials. The lecture aims to show, by means of accessible examples, that interesting research problems arise from asking questions about the characteristic properties satisfied by classical orthogonal polynomials and their extensions or generalisations. We review properties that have been used to derive upper and lower bounds for the extreme zeros of orthogonal polynomials. Topics to be covered include Markov's theorem on monotonicity of zeros and its generalisations, the proof of a conjecture by Askey and its extensions, interlacing properties of zeros, Sturm's comparison theorem and convexity of zeros.

Plenary Talk 21: *Orthogonal polynomials and Painlevé equations*

Thursday 11 October, 08:30-09:30
Speaker: *Walter Van Assche*

Orthogonal polynomials on the real line always satisfy a three-term recurrence relation which for the monic polynomials is

$$xP_n(x) = P_{n+1}(x) + b_n P_n(x) + a_n^2 P_{n-1}(x).$$

An important problem is to find the recurrence coefficients $(a_n^2, b_n)_n$ when the orthogonality measure (or weight) is given. For classical orthogonal polynomials these recurrence coefficients can be computed fairly easy, but when we take semiclassical weights, one finds nonlinear recurrence relations for the recurrence coefficients. These nonlinear recurrence relations turn out to be discrete Painlevé equations. We will give a few examples of this feature. If the weight is of the form $w_t(x) = e^{xt}w(x)$, with w a fixed weight and $t \in \mathbb{R}$, then the recurrence coefficients $a_n^2(t)$ and $b_n(t)$ depend on t and they satisfy an infinite system of differential equations known as the Toda lattice. Combining these Toda equations with the discrete Painlevé equations then gives a nonlinear differential equation for the recurrence coefficients, which is often one of the six Painlevé differential equations. Examples will be given and we will try to explain this connection between orthogonal polynomials and Painlevé equations.

Plenary Talk 22: *Properties of certain classes of semiclassical orthogonal polynomials*

Thursday 11 October, 09:30-10:30
Speaker: *Kerstin Jordaan*

In this lecture we discuss properties of orthogonal polynomials for weights which are semiclassical perturbations of classical orthogonality weights. We use the moments, together with the connection between orthogonal polynomials and Painlevé equations to obtain explicit expressions for the recurrence coefficients of polynomials associated with a semiclassical Laguerre and a generalized Freud weight. We derive a second-order linear ordinary differential equation and a differential-difference equation satisfied by the generalized Freud polynomials and analyze the asymptotic behavior of the polynomials in two different contexts. We show that unique, positive solutions of the nonlinear difference equation satisfied by the recurrence coefficients exist for all real values of the parameter involved in the semiclassical perturbation but that these solutions are very sensitive to the initial conditions. We also prove properties of the zeros of semiclassical Laguerre and generalized Freud polynomials.

Plenary Talk 23: *Quantum state revivals, graphs and orthogonal polynomials*

Thursday 11 October, 11:00-12:00
Speaker: *Luc Vinet*

This lecture will describe how certain features of quantum transport along spin chains can be enabled. With an eye to extension to higher dimensional lattices, it will also discuss connections with quantum walks on graphs of the Hamming scheme and one of its generalizations. Some univariate and multivariate orthogonal polynomials will be seen to play a central role.

Plenary Talk 24: *Some characterization problems related to Sheffer polynomial sets*

Thursday 11 October, 12:00-13:00
Speaker: *Hamza Chaggara*

The polynomial sequences of Sheffer type $\{P_n\}_{n\geq 0}$ are defined by the generating function:

$$G(x,t) = \sum_{n=0}^{\infty} \frac{P_n(x)}{n!} t^n = A(t)e^{xC(t)}, \tag{3}$$

where $A(t) = \sum_{n=0}^{\infty} a_n t^n$, $a_0 \neq 0$ (resp. $C(t) = \sum_{n=1}^{\infty} c_n t^n$, $c_1 \neq 0$).
d-orthogonal polynomial sets (d being a non-negative integer) are polynomials satisfying one standard $(d+1)$-order recurrence relation.
We are interested, in this talk, with Sheffer type sequences when they satisfy additional properties. Indeed

- Orthogonal Polynomials of Sheffer type.
- d-orthogonal polynomials of Sheffer type.
- Symmetric d-orthogonal polynomials of Sheffer type.

Plenary Talk 25: *Unique positive solution for an alternative discrete Painlevé-I equation*

Thursday 11 October, 14:30-15:30
Speaker: *Ana Loureiro*

The system of two nonlinear equations

$$a_n + a_{n+1} = b_n^2 - t,$$
$$a_n(b_n + b_{n-1}) = n,$$

is known as the *alternative discrete Painlevé I equation* (alt-dP_I). These equations arise, for example, when one wants to compute the recurrence coefficients of orthogonal polynomials (as well as multiple orthogonal polynomials) with an exponential cubic weight. They also give relations between solutions of the second Painlevé equation P_{II}

$$\frac{d^2w}{dt^2} = 2w^3 + tw + \alpha, \tag{4}$$

for different values of the parameter α, after a scaling on the variable. We will show that the *alternative discrete Painlevé I equation* has a unique solution that remains positive for all $n \geq 0$, while identifying this positive solution in terms of a special-function solutions of P_{II} involving only the Airy function $\mathrm{Ai}(t)$. Such a solution has the property that it remains positive for all $n \geq 0$ and $t \geq 0$.

Contributed Talk 27: *Classical discrete d-orthogonal polynomials*

Thursday 11 October, 17:00-17:30
Speaker: *Naoures Ayadi*

In this talk, we give a characterization of classical d-orthogonal polynomial sets by a Pearson type matrix equation satisfied by the associated d-dimensional function vector. We illustrate by some examples.

Contributed Talk 28: *Modified classical orthogonal polynomials satisfying Heun's differential equation*

Thursday 11 October, 17:30-18:00
Speaker: *François Ndayiragije*

We consider special families of orthogonal polynomials satisfying differential equations. Besides known hypergeometric cases, we look especially for ***Heun***'s differential equations.We show that such equations are satisfied by orthogonal polynomials related to some classical weight functions modified by Dirac weights.
Keywords: Orthogonal polynomials, Padé approximation, Heun's differential equation.

Plenary Talk 29: *Some systems of multivariate orthogonal polynomials*

Friday 12 October, 08:30-09:30
Speaker: *Iván Area*

In this talk we review some recent results concerning multivariate orthogonal polynomials. The main tool is the partial differential, difference, q-difference or divided-difference equation they satisfy. From these equations satisfied by the orthogonal polynomial sequences, the equation satisfied by any derivative (difference, q-difference or divided-difference) of the polynomials is obtained. Moreover, we obtain explicitly the matrix coefficients appearing in the three-term recurrence relations satisfied by any multivariate orthogonal polynomial solution of the equation. Some examples show the importance of this approach, namely bivariate Appell polynomials for the continuous case, bivariate Hahn polynomials for the discrete case, bivariate big q-Jacobi polynomials for the q-case, and bivariate Racah, Wilson, q-Racah, and Askey-Wilson polynomials for equations involving divided-difference operators. The main difficulties when moving from the univariate case to the multivariate case(s) will be analyzed in detail.

Plenary Talk 30: ***Tridiagonalization and Heun operators***

Friday 12 October, 09:30-10:30
Speaker: *Luc Vinet*

The standard Heun equation is a second order differential equation with four regular singularities. The defining operator maps polynomials of degree n into polynomials of degree n+1. It will be shown that this Heun operator can be obtained by performing the tridiagonalization of the hypergeometric operator. This naturally implies that the Heun operator acts tridiagonally on Jacobi polynomials.
Extension of the Heun equation on discrete lattices will be obtained through the concept of algebraic Heun operators (AHO) associated to bispectral situations, AHO being bilinear expressions in the operators defining the bispectral problem. The Heun operators on uniform and exponential lattices associated to Hahn and big q-Jacobi polynomials will be constructed and studied. It will be seen in particular that the AHOs that are found coincide like in the standard case, with the most general operators on the corresponding lattices that raise the degrees by one.

Plenary Talk 31: ***On difference equations for orthogonal polynomials on special nonuniform lattices***

Friday 12 October, 11:00-12:00
Speaker: *Mama Foupouagnigni*

In this talk, we derive an appropriate polynomial basis which is then used to: 1) provide the functional approach of the characterization theorem for classical orthogonal polynomials on nonuniform lattice; 2) providing algorithmic method to find explicit solution to holonomic divided-difference equation and 3) the derivation of some structure relations for classical orthogonal polynomials on nonuniform lattices such the determination of the coefficients of the connection and the linearisation problems involving classical orthogonal polynomials on nonuniform lattices.
This is a joint work with: Wolfram Koepf (University of Kassel, Germany), Salifou Mboutngam (University of Maroua, Cameroon), Maurice Kenfack-Nangho (University of Dschang, Cameroon), Daniel Duviol Tcheutia (University of Kassel, Germany) and Patrick Njionou Sadjang (University of Douala, Cameroon).

Plenary Talk 32: ***Orthogonal polynomials and computer algebra***

Friday 12 October, 12:00-13:00
Speaker: *Wolfram Koepf*

Classical orthogonal polynomials of the Askey-Wilson scheme have extremely many different properties, e.g. satisfying differential equations, recurrence equations, having hypergeometric representations, Rodrigues formulas, generating functions, moment representations etc. Using computer algebra it is possible to switch between one representation and another algorithmically. Such algorithms will be discussed and implementations are presented using Maple.