# Properties and applications of the zeros of classical continuous orthogonal polynomials

Alta Jooste and D.D. Tcheutia

UNIKASSEL VERSITÄT

AIMS–Volkswagen Stiftung Workshop on Introduction to Orthogonal Polynomials and Applications
Douala
October 5-12, 2018

# Table of Contents

# The zeros of the Tchebyshev polynomials of the first kind

We consider the Tchebyshev polynomials of the first kind defined by

$$T_n(x) = \cos(n \arccos x).$$

It was shown that

- The sequence $\{T_n\}_{n\geq 0}$ is orthogonal on $(-1, 1)$ with respect to the weight function $w(x) = 1 - x^2$.
- $$x_{n,k} = \cos \frac{2(n-k)-1}{2n}\pi, \; k = 0, 1, \dots, n-1$$
  are the zeros of $T_n(x)$ such that
  $$-1 < x_{n,0} < x_{n,1} < \dots < x_{n,n-2} < x_{n,n-1} < 1.$$

That means $T_n(x)$ has exactly $n$ simple zeros in the interval $(-1, 1)$ of orthogonality.

# $p_n(x)$ has exactly $n$ simple zeros in $(a, b)$

## Theorem

If $\{p_n\}_{n=0}^{\infty}$ is a sequence of polynomials orthogonal on $(a, b)$, with respect to the weight function $w(x)$, then the polynomial $p_n(x)$ has exactly $n$ simple zeros in $(a, b)$.

# Proof

Since $\deg(p_n(x)) = n$ the polynomial $p_n$ has at most $n$ real zeros. Suppose that $p_n(x)$ has $m \leq n$ distinct real zeros $x_1, x_2, \dots, x_m$ in $(a, b)$ of odd order (or multiplicity). We want to show that $m = n$. We consider the polynomial

$$p_n(x)q_m(x), \text{ with } q_m(x) = (x - x_1)(x - x_2)\cdots(x - x_m).$$

The zeros of $p_n(x)q_m(x)$ are of even order, therefore $p_n(x)q_m(x)$ does not change sign in $(a, b)$ and then

$$\int_a^b w(x)p_n(x)q_m(x)dx \neq 0.$$

Since $\{p_n\}_{n=0}^{\infty}$ is a sequence of orthogonal polynomials, this integral equals zero if $m < n$. Hence $m = n$, which implies that $p_n(x)$ has $n$ distinct real zeros of odd order in $(a, b)$. This proves that all $n$ zeros are distinct and simple.

# The zeros of $p_n(x)$ and $p_{n+1}(x)$ separate each other

The zeros of $T_n(x)$ are

$$x_{n,k} = \cos \frac{2(n-k)-1}{2n}\pi.$$

Can $T_n(x)$ and $T_{n+1}(x)$ have a common zero?

**Theorem**

If $\{p_n(x)\}_{n=0}^{\infty}$ is a sequence of orthogonal polynomials on the interval $(a, b)$ with respect to the weight function $w(x)$, then the zeros of $p_n(x)$ and $p_{n+1}(x)$ separate each other.

# Proof

Without loss of generality, we suppose $\{p_n\}_n$ is a monic sequence since $p_n = k_n x^n + \dots$ and $\frac{1}{k_n} p_n = x^n + \dots$ have the same zeros.
We substitute $x$, respectively, by $x_{n+1,k}$ and $x_{n+1,k+1}$ in the confluent form of the Darboux-Christoffel formula

$$\sum_{k=0}^{n} \frac{p_k(x)^2}{h_k} = \frac{1}{h_n}\left(p'_{n+1}(x)p_n(x) - p_{n+1}(x)p'_n(x)\right)$$

to get

$$p'_{n+1}(x_{n+1,k})p_n(x_{n+1,k}) > 0 \text{ and } p'_{n+1}(x_{n+1,k+1})p_n(x_{n+1,k+1}) > 0.$$

Recall: (**Rolle's theorem**) If $f$ is a continuous and differentiable function on $[a,b]$ such that $f(a) = f(b)$, there exists $x_0 \in (a,b)$ such that $f'(x_0) = 0$.

# Proof

$p_{n+1}$ is continuous, $p_{n+1}(x_{n+1,k}) = p_{n+1}(x_{n+1,k+1}) = 0$ and from Rolle's theorem, $p'_{n+1}$ has exactly one zero in $(x_{n+1,k}, x_{n+1,k+1})$ (since the zeros of$p_{n+1}$ are simple). It follows that $p'_{n+1}(x_{n+1,k})$ and $p'_{n+1}(x_{n+1,k+1})$ have opposite signs. From

$$p'_{n+1}(x_{n+1,k})p_n(x_{n+1,k}) > 0 \text{ and } p'_{n+1}(x_{n+1,k+1})p_n(x_{n+1,k+1}) > 0,$$

we deduce that $p_n(x_{n+1,k})$ and $p_n(x_{n+1,k+1})$ also have opposite signs. Since $p_n$ is continuous, $p_n$ has at least one zero in $(x_{n+1,k}, x_{n+1,k+1})$. This zero is $x_{n,k}$ since we have $n$ intervals of type $(x_{n+1,k}, x_{n+1,k+1})$. Conclusion,

$$a < x_{n+1,1} < x_{n,1} < x_{n+1,2} < x_{n,2} < \cdots < x_{n+1,n} < x_{n,n} < x_{n+1,n+1} < b.$$

# Example of the Jacobi polynomials

Consider the Jacobi polynomials:

$$P_n^{\alpha,\beta}(x) = \frac{(\alpha+1)_n}{n!}\ {}_2F_1\left(\begin{array}{c} -n, n+\alpha+\beta+1 \\ \alpha+1 \end{array}\middle| \frac{1-x}{2}\right).$$

For $\alpha, \beta > -1$, they are orthogonal with respect to the weight function $w(x) = (1-x)^\alpha(1+x)^\beta$ on the interval $(-1, 1)$.

Mathematica

# Lagrange interpolation

The Gauss quadrature formula is of use for the approximation of integrals $\int_a^b f(x)w(x)dx$ in numerical analysis. In this section, we show how to use zeros of orthogonal polynomials in the computation of integrals.
If $f$ is a continuous function in $(a, b)$ and $x_1 < x_2 < \cdots < x_n$ are $n$ distinct points in $(a, b)$, then there exists exactly one polynomial $L$, with $\deg(L) \leq n - 1$ such that $L(x_j) = f(x_j)$ for all $j = 1, 2, \ldots, n$. This polynomial $L$ can easily be found by using Lagrange interpolation. Define

$$p(x) = (x - x_1)(x - x_2) \cdots (x - x_n)$$

and consider the Lagrange interpolation polynomial

$$L(x) = \sum_{k=1}^{n} f(x_k) \frac{p(x)}{(x - x_k)p'(x_k)} = \sum_{k=1}^{n} f(x_k) \prod_{i=1, i \neq k}^{n} \frac{(x - x_i)}{(x_k - x_i)}.$$

# $f$ is a polynomial of degree $\leq 2n-1$

Let $\{p_n\}_{n=0}^{\infty}$ be a sequence of polynomials orthogonal on $(a,b)$, with respect to the weight function $w(x)$, with $n$ distinct real zeros $x_1 < x_2 < \cdots < x_n$. If $f$ is a polynomial of degree $\leq 2n-1$, then $f(x) - L(x)$ is a polynomial of degree $\leq 2n-1$ with at least the zeros $x_1 < x_2 < \cdots < x_n$. Now define

$$f(x) = L(x) + r(x)p_n(x),$$

where $r(x)$ is a polynomial of degree $\leq n-1$. This can also be written as

$$f(x) = \sum_{k=1}^{n} f(x_k)\frac{p_n(x)}{(x-x_k)p_n'(x_k)} + r(x)p_n(x).$$

This implies that

$$\int_a^b f(x)w(x)dx = \sum_{k=1}^{n} f(x_k)\int_a^b \frac{p_n(x)}{(x-x_k)p_n'(x_k)}w(x)dx + \int_a^b p_n(x)r(x)w(x)dx$$

# $f$ is a polynomial of degree $\leq 2n-1$

Since $\deg(r(x)) \leq n-1$, the latter integral equals zero, due to orthogonality. We thus have

$$\int_a^b f(x)w(x)dx = \sum_{k=1}^{n} \lambda_{n,k} f(x_k)$$

with

$$\lambda_{n,k} := \int_a^b \frac{p_n(x)}{(x-x_k)p_n'(x_k)} w(x)dx, \ k = 1, 2, \ldots, n.$$

This is the Gauss quadrature formula which gives the value of the integral in the case that $f$ is a polynomial of degree $\leq 2n-1$ if the values of $f(x_k)$ is known for the $n$ zeros $x_1 < x_2 < \cdots < x_n$ of the polynomial $p_n$.

# $f$ is not a polynomial of degree $\leq 2n-1$

If $f$ is not a polynomial of degree $\leq 2n-1$, this leads to an approximation of the integral

$$\int_a^b f(x)w(x)dx \approx \sum_{k=1}^{n} \lambda_{n,k} f(x_k)$$

with

$$\lambda_{n,k} := \int_a^b \frac{p_n(x)}{(x-x_k)p_n'(x_k)} w(x)dx, \; k = 1, 2, \ldots, n.$$

The coefficients $\{\lambda_{n,k}\}_{k=1}^{n}$ are called the Christoffel numbers. One can show that

$$\lambda_{n,k} = -\frac{k_{n+1}}{k_n} \frac{h_n}{p_{n+1}(x_{n,k})p_n'(x_{n,k})},$$

where $k_n$ is the leading coefficient of $p_n$.

# The Christoffel numbers are all positive

The Christoffel numbers are all positive. This can be shown as follows. We have

$$\lambda_{n,k} = \int_a^b l_{n,k}(x)w(x)dx, \text{ with } l_{n,k} = \frac{p_n(x)}{(x-x_k)p_n'(x_k)}, \; k = 1, 2, \ldots, n.$$

Then $l_{n,k}^2 - l_{n,k}$ is a polynomial of degree $\leq 2n-2$, which vanishes at the zeros of $p_n$, namely $x_{n,1}, x_{n,2}, \ldots, x_{n,n}$. We can write

$$l_{n,k}(x)^2 - l_{n,k}(x) = p_n(x)q(x),$$

where $q(x)$ is a polynomial of degree $\leq n-2$. Then

$$\int_a^b \left(l_{n,k}(x)^2 - l_{n,k}(x)\right) w(x)dx = \int_a^b p_n(x)q(x)w(x)dx = 0,$$

since $\deg(q(x)) \leq n-2$, consequently

$$\lambda_{n,k} = \int_a^b l_{n,k}(x)w(x)dx = \int_a^b l_{n,k}(x)^2 w(x)dx > 0.$$

# Exercise

Consider

$$\int_{-1}^{1} \frac{dx}{x+3} = \int_{a}^{b} f(x)w(x)dx,$$

where $f(x) = \frac{1}{x+3}$, $a = -1$, $b = 1$, $w(x) = 1$. We want to use the zeros of the second degree Legendre polynomial

$$L_n(x) = {}_2F_1\left(\begin{array}{c} -n, n+1 \\ 1 \end{array} \middle| \frac{1-x}{2}\right),$$

orthogonal with respect to $w(x) = 1$ on $[-1, 1]$, i.e.,

$$L_2(x) = \frac{1}{2}\left(-1 + 3x^2\right),$$

to estimate this integral.

# Solution

The zeros of $L_2(x)$ are $x_1 = \frac{1}{\sqrt{3}}$ and $x_2 = -\frac{1}{\sqrt{3}}$.
We have

$$\int_{-1}^{1} \frac{dx}{x+3} \approx \lambda_{2,1} f(x_1) + \lambda_{2,2} f(x_2) = \frac{\lambda_{2,1}}{x_1+3} + \frac{\lambda_{2,2}}{x_2+3},$$

where

$$\lambda_{2,k} = \int_{-1}^{1} \frac{L_2(x)}{(x-x_k)L_2'(x_k)} w(x)dx = \int_{-1}^{1} \frac{\frac{1}{2}\left(-1+3x^2\right)}{(x-x_k)3x_k} dx, k = 1, 2,$$

i.e.,$\lambda_{2,1} = \lambda_{2,2} = 1$ and

$$\int_{-1}^{1} \frac{dx}{x+3} = \frac{1}{\left(\frac{1}{\sqrt{3}}+3\right)} + \frac{1}{\left(-\frac{1}{\sqrt{3}}+3\right)} = 0.69231.$$

Mathematica

# Exercise

The Chebyshev polynomials of the second kind defined by

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \; x = \cos\theta, \; 0 < \theta < \pi.$$

For this family,

$$k_n = 2^n, \; h_n = \int_{-1}^{1} U_n(x)U_n(x)(1-x^2)^{\frac{1}{2}}dx = \frac{\pi}{2}$$

and the zeros of $U_n$ in increasing order are $x_{n,k} = \cos\theta_{n,k} = \cos\frac{n-k+1}{n+1}\pi, \; k = 1, 2, \ldots, n.$ We want to show that

$$\int_{-1}^{1} \sqrt{1-x^2}x^4 dx = \frac{\pi}{16}.$$

# Exercise

We show using

$$\lambda_{n,k} = -\frac{k_{n+1}}{k_n}\frac{h_n}{p_{n+1}(x_{n,k})p_n'(x_{n,k})},$$

that $\lambda_{n,k} = \frac{\pi \sin^2 \theta_{n,k}}{n+1}$.

1. Show that in the case of this integral, $n = 3$.
2. Compute the zeros of $U_3(x)$ and show that

$$\lambda_{3,1} = \frac{1}{4}\frac{\pi}{2},\ \lambda_{3,2} = \frac{1}{4}\pi,\ \lambda_{3,3} = \frac{1}{4}\frac{\pi}{2}.$$

3. Deduce that

$$\int_{-1}^{1} \sqrt{1-x^2}x^4 dx = \frac{\pi}{16}.$$

# Hermite interpolating polynomial

Consider $n+1$ distinct real numbers $\{z_{n,i}\}_{i=0}^{n}$ and let $f \in \mathcal{C}^1[-1,1]$, then there exists a unique polynomial $Q_{2n+1}(f;z)$ of degree at most $2n+1$ such that

$$\begin{cases} Q_{2n+1}(f;z_{n,i}) = f(z_{n,i}), & i = 0,1,\ldots,n, \\ Q'_{2n+1}(f;z_{n,i}) = f'(z_{n,i}), & i = 0,1,\ldots,n. \end{cases}$$

The polynomial $Q_{2n+1}(f;z)$ is called the Hermite interpolating polynomial for $f$.

There exists a basis $\{A_{n,i}(z), B_{n,i}(z),\ i = 0,1,2,\ldots,n\}$ $(A_{n,i}(z), B_{n,i}(z) \in \mathbb{R}_{2n+1}[z],\ \text{ for each } i = 0,1,\ldots,n)$ such that

$$Q_{2n+1}(f;z) = \sum_{i=0}^{n} f(z_{n,i})A_{n,i}(z) + \sum_{i=0}^{n} f'(z_{n,i})B_{n,i}(z).$$

# Hermite interpolating polynomial

For $f(z) = 1$, $z \in [-1, 1]$,

$$1 = Q_{2n+1}(f; z) = \sum_{i=0}^{n} f(z_{n,i}) A_{n,i}(z) = \sum_{i=0}^{n} A_{n,i}(z),$$

that is

$$\sum_{i=0}^{n} A_{n,i}(z) = 1.$$

Looking for necessary conditions on $A_{n,i}(z)$ and $B_{n,i}(z)$ for the condition

$$\begin{cases} Q_{2n+1}(f; z_{n,i}) = f(z_{n,i}), & i = 0, 1, \ldots, n, \\ Q'_{2n+1}(f; z_{n,i}) = f'(z_{n,i}), & i = 0, 1, \ldots, n. \end{cases}$$

to be satisfied, we get

$$\begin{cases} A_{n,i}(z_{n,j}) = \delta_{ij} \text{ and } B_{n,i}(z_{n,j}) = 0, & i, j = 0, 1, \ldots, n \\ A'_{n,i}(z_{n,j}) = 0 \text{ and } B'_i(z_{n,j}) = \delta_{ij}, & i, j = 0, 1, \ldots, n. \end{cases}$$

# Hermite interpolating polynomial

Let $\{l_{n,i}(z),\ i=0,1,\ldots,n\}$ be the Lagrange basis polynomials in $\mathbb{R}_n[z]$ (the set of polynomials of degree at most $n$ with real coefficients) defined by

$$l_{n,k}(z)=\prod_{\substack{i=0\\ i\neq k}}^{n}\left(\frac{z-z_{n,i}}{z_{n,k}-z_{n,i}}\right),\quad k=0,1,2,\ldots,n,$$

then the Hermite basis polynomials $\{A_{n,i}(z),B_{n,i}(z),\ i=0,1,\ldots,n\}$ in $\mathbb{R}_{2n+1}[z]$ are given by:

$$\begin{aligned}A_{n,i}(z)&=(1-2(z-z_{n,i})l'_{n,i}(z_{n,i}))l^2_{n,i}(z),\ i=0,1,\ldots,n,\\ B_{n,i}(z)&=(z-z_{n,i})l^2_{n,i}(z),\ i=0,1,\ldots,n.\end{aligned}$$

This means that the Hermite interpolation polynomials are given in terms of the Lagrange basis polynomials as

$$Q_{2n+1}(f;z)=\sum_{i=0}^{n}f(z_{n,i})\Big(1-2(z-z_{n,i})l'_{n,i}(z_{n,i})\Big)l^2_{n,i}(z)+\sum_{i=0}^{n}f'(z_{n,i})(z-z_{n,i})l^2_{n,i}(z$$

# Newton polynomial

We can show by direct calculus that for the Newton polynomial

$$N_k(z) = \begin{cases} 1, & \text{if } k = 0 \\ (z - z_{n+1,0})(z - z_{n+1,1})\cdots(z - z_{n+1,k-1}), & \text{if } 1 \le k \le n+1, \end{cases}$$

we have

$$l_{n,i}(z) = \frac{N_{n+1}(z)}{(z - z_{n+1,i})N'_{n+1}(z_{n+1,i})} \text{ and } 2l'_{n,i}(z_{n+1,i}) = \frac{N''_{n+1}(z_{n+1,i})}{N'_{n+1}(z_{n+1,i})}.$$

Let $0 < \theta < \pi$, set $z = \cos\theta$ and define for $n = 0, 1, \ldots$

$$T_n(z) = \cos(n\theta).$$

The roots $z_{n,k}$ of $T_n(z)$ in increasing order are given by

$$z_{n,k} = \cos\theta_{n,k}, \text{ with } \theta_{n,k} = \frac{(2(n-k)-1)\pi}{2n}, \; k = 0, 1, \ldots, n-1.$$

# Main result

For this polynomial family, we have

$$A_{1,n,i}(z) = (1 - zz_{n+1,i})\left(\frac{T_{n+1}(z)}{(n+1)(z - z_{n+1,i})}\right)^2.$$

Let us now state and prove the interpolation and approximation result for the first kind Chebyshev polynomials.

## Theorem

*Let $f \in \mathcal{C}[-1,1]$, the Hermite interpolation polynomials $Q_{2n+1}(f;z)$ (at the zeros $z_{n+1,k}$, $k = 0, 1, \ldots, n$, of the Chebyshev polynomials $T_{n+1}(z)$ which satisfies*

$$\begin{cases} Q_{2n+1}(f; z_{n+1,i}) = f(z_{n+1,i}), & i = 0, 1, \ldots, n, \\ Q'_{2n+1}(f; z_{n+1,i}) = 0, & i = 0, 1, \ldots, n, \end{cases}$$

*converge uniformly on $[-1,1]$ to $f$.*

Maple

# proof

$z, z_{n+1,i} \in [-1,1] \Rightarrow z z_{n+1,i} \in [-1,1]$ and then $1 - z z_{n+1,i} \geq 0$, thus $A_{1,n,i}(z) \geq 0$. From $\sum_{i=0}^{n} A_{n,i}(z) = 1$, we have $f(z) = \sum_{i=0}^{n} f(z) A_{1,n,i}(z)$. Let $\epsilon > 0$ and $z \in [-1,1]$. We want to show that

$$\exists\, N_\epsilon \in \mathbb{N} \text{ such that } \forall n \geq N_\epsilon,\ |f(z) - Q_{1,2n+1}(f;z)| < \epsilon.$$

$$\begin{aligned}|f(z) - Q_{1,2n+1}(f;z)| &= \left|\sum_{i=0}^{n} (f(z) - f(z_{n+1,i})) A_{1,n,i}(z)\right| \\ &\leq \sum_{i=0}^{n} |f(z) - f(z_{n+1,i})| A_{1,n,i}(z).\end{aligned}$$

$f$ continuous on $[-1,1]$ and $[-1,1]$ is compact implies $f$ is uniformly continuous on $[-1,1]$. That is
$\exists \delta_\epsilon > 0$ such that $\forall\ x, y \in [-1,1]$, $|x-y| < \delta_\epsilon$ implies $|f(x) - f(y)| < \epsilon$.
Let $I_{n,\epsilon,z} := \{i \in \{0,1,\ldots,n\}\ :\ |z - z_{n+1,i}| < \delta_\epsilon\}$ and $J_{n,\epsilon,z} := \{i \in \{0,1,\ldots,n\}\ :\ |z - z_{n+1,i}| \geq \delta_\epsilon\}$. Then $I_{n,\epsilon,z} \cup J_{n,\epsilon,z} = \{0,1,\ldots,n\}$ and $I_{n,\epsilon,z} \cap J_{n,\epsilon,z} = \emptyset$,

# proof

$$\sum_{i=0}^{n} |f(z) - f(z_{n+1,i})| A_{1,n,i}(z) = \sum_{i \in I_{n,\epsilon,z}} |f(z) - f(z_{n+1,i})| A_{1,n,i}(z) + \sum_{i \in J_{n,\epsilon,z}} |f(z) - f(z_{n+1,i})| A_{1,n,i}(z).$$

If $i \in I_{n,\epsilon,z}$, $|z - z_{n+1,i}| < \delta_\epsilon$ and then $|f(z) - f(z_{n+1,i})| < \frac{\epsilon}{2}$ such that

$$\sum_{i \in I_{n,\epsilon,z}} |f(z) - f(z_{n+1,i})| A_{1,n,i}(z) < \frac{\epsilon}{2} \sum_{i \in I_{n,\epsilon,z}} A_{1,n,i}(z) < \frac{\epsilon}{2} \sum_{i=0}^{n} A_{1,n,i}(z) = \frac{\epsilon}{2},$$

where we use respectively the fact that $A_{1,n,i}(z) \geq 0$, $I_{n,\epsilon,z} \subset \{0, 1, \ldots, n\}$ and $\sum_{i=0}^{n} A_{1,n,i}(z) = 1$.
$f \in \mathcal{C}[-1, 1]$ implies $f$ is bounded. That is, $\exists\, M > 0$ suct that $|f(z)| < M$, $\forall\, z \in [-1, 1]$. Therefore $|f(z) - f(z_{n+1,i})| \leq |f(z)| + |f(z_{n+1,i})| \leq 2M.$

# proof

So

$$\sum_{i\in J_{n,\epsilon,z}} |f(z)-f(z_{n+1,i})|A_{1,n,i}(z) \leq 2M \sum_{i\in J_{n,\epsilon,z}} A_{1,n,i}(z).$$

We have

$$i \in J_{n,\epsilon,z} \Rightarrow |z - z_{n+1,i}| \geq \delta_\epsilon \Leftrightarrow \frac{1}{|z-z_{n+1,i}|^2} \leq \frac{1}{\delta_\epsilon^2},$$

$$|z| \leq 1,\ |z_{n+1,i}| \leq 1 \Rightarrow |zz_{n+1,i}| \leq 1 \Leftrightarrow -1 \leq zz_{n+1,i} \leq 1$$
$$\Leftrightarrow 0 \leq 1 - zz_{n+1,i} \leq 2 \Rightarrow 1 - zz_{n+1,i} \leq 2,$$

and

$$|T_{n+1}(z)| = |\cos(n+1)\theta| \leq 1.$$

This leads to

$$\sum_{i\in J_{n,\epsilon,z}} |f(z)-f(z_{n+1,i})|A_{1,n,i}(z) \leq 2M \sum_{i\in J_{n,\epsilon,z}} (1-zz_{n+1,i})\left(\frac{T_{n+1}(z)}{(n+1)(z-z_{n+1,i})}\right)^2$$

# proof

Since

$$\lim_{n\longrightarrow\infty}\left(\frac{4M}{\delta_\epsilon^2(n+1)}\right)=0,$$

then for $\frac{\epsilon}{2}>0$

$$\exists\, N_\epsilon\in\mathbb{N} \text{ such that } \forall\, n\geq N_\epsilon,\ \frac{4M}{\delta_\epsilon^2(n+1)}<\frac{\epsilon}{2}$$

$$\Rightarrow \sum_{i\in J_{n,\epsilon,z}}|f(z)-f(z_{n+1,i})|A_{1,n,i}(z)<\frac{\epsilon}{2}.$$

In conclusion, we have $\forall\, n\geq N_\epsilon$,

$$\sum_{i=0}^{n}|f(z)-f(z_{n+1,i})|A_{1,n,i}(z)\leq\sum_{i\in I_{n,\epsilon,z}}|f(z)-f(z_{n+1,i})|A_{1,n,i}(z)$$

$$+\sum_{i\in J_{n,\epsilon,z}}|f(z)-f(z_{n+1,i})|A_{n,i}(z)\leq\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.$$

Hence $(Q_{1,2n+1}(f;z))_n$ converges uniformly to $f$ on $[-1,1]$.

# Thank for your attention