On the Solutions of Holonomic Third-order Linear Irreducible Differential Equations in Terms of Hypergeometric Functions

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Plan

1. Introduction
2. Previous works
3. My work

Holonomic differential equations

We consider a differential equation of type

$$a_n(x)\, y^{(n)}(x) + a_{n-1}(x)\, y^{(n-1)}(x) + \ldots + a_0(x)\, y(x) = 0,\ n \in \mathbb{N}.$$

When the coefficients $a_i(x), i = 0, \ldots, n$ are polynomials of the variable x, the differential equation is said to be holonomic.

Remark

For a given homogeneous linear differential equation with rational function coefficients, one can multiply by their common denominator and get a holonomic differential equation with the same space of solutions.

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Differential operators

Every holonomic differential equation

$$a_n(x)\, y^{(n)}(x) + a_{n-1}(x)\, y^{(n-1)}(x) + \ldots + a_0(x)\, y(x) = 0,$$

with $a_i(x) \in \mathbb{Q}[x]$, corresponds to a differential operator L given by

$$\mathrm{L} = \sum_{i=0}^{n} a_i(x) D_x^i, \qquad (1)$$

and vice versa. Hence,

Remark

By the solutions of a differential operator L we mean the solutions y of the holonomic differential equation $\mathrm{L}y = 0$.

Differential operators

Definition

Let

$$\mathrm{L} = \sum_{i=0}^{n} a_i(x) D_x^i,$$

and $n_0 = \max\{i = 0, 1, \ldots, n \quad a_i(x) \neq 0\}$. n_0 is called the order of L, and $a_{n_0}(x)$ its leading coefficient.

The ring of differential operators with coefficients in $\mathrm{K} = \mathbb{Q}[x]$ is denoted by $\mathrm{K}[D_x]$. In our context, $D_x := \frac{d}{dx}$.

First-order holonomic differential equations

First-order holonomic differential equations are of type

$$a_1(x)\, y' + a_0(x)\, y = 0 \quad \text{with } a_0(x), a_1(x) \in \mathbb{Q}[x], \quad a_1(x) \neq 0.$$

That means, non-zero solutions satisfy

$$\frac{y'}{y} = -\frac{a_0(x)}{a_1(x)}, \tag{2}$$

and they can be easily computed in the form

$$y(x) = c \cdot \exp\left(\int -\frac{a_0(x)}{a_1(x)}\, dx\right) \quad \text{with } c \in \mathbb{R}. \tag{3}$$

Those solutions are called hyperexponential functions.

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Example of first-order holonomic differential equations

Let us consider the following differential equation

$$(x^2+1)y' - xy = 0.$$

Non-zero solutions are

$$\begin{aligned} y(x) &= c \cdot \exp\left(\int -\frac{-x}{x^2+1}\, dx\right) = c \cdot \exp\left(\frac{1}{2} \ln\left(x^2+1\right)\right) \\ &= c \cdot \sqrt{x^2+1} \quad \text{with } c \in \mathbb{R}. \end{aligned}$$

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Second-order holonomic differential equations

Second-order holonomic differential equations are of the form

$$(eq):\ a_2(x)\,y'' + a_1(x)\,y' + a_0(x)\,y = 0$$

with $a_0(x), a_1(x), a_2(x) \in \mathbb{Q}[x]$, $a_2(x) \neq 0$.

If the **operator associated to** (eq) **is reducible**, solutions of (eq) can be easily computed, since we know how to solve first-order holonomic differential equations. To check the reducibility of an operator, we can use some known algorithms like Beke's algorithm. This algorithm was extended by Mark van Hoeij in his PhD thesis on factorization of linear differential operators (1996).

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Example of reducible second-order holonomic differential equations

Let us consider the following differential equation

$$(E2): \quad (x+7)y'' - x(x+7)y' + xy = 0. \tag{4}$$

Its associated operator can be factorized as follows:

$$(x+7)D_x^2 - x(x+7)D_x + x = (D_x - x)\cdot((x+7)D_x - 1)$$

Hence, solving $(x+7)y' - y = 0$, we get a solution of $(E2)$ given by

$$y(x) = \exp\left(\int \frac{1}{x+7}\,dx\right) = \exp\left(\ln\left(x+7\right)\right) = x+7.$$

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Second-order holonomic differential equations

If **the differential operator associated to**

$$(eq): \; a_2(x)\, y'' + a_1(x)\, y' + a_0(x)\, y = 0$$

is irreducible, it is more difficult to find solutions of (eq). There are some algorithms which try to find them in some particular forms. Kovacic's algorithm (which finds Liouvillian solutions) is an example. Some complete algorithms which solve (eq) in terms of some special functions are given by Ruben Debeerst, Mark Van Hoeij, Wolfram Koepf and Quan Yuan.

Special functions type solutions

The complete algorithm to solve irreducible second-order differential equations in terms of special functions tries to find all solutions of the type

$$\exp\left(\int r\,dx\right)\left(r_0 S(f(x)) + r_1\left(S(f(x))\right)'\right)$$

where S is the special function that we want to solve in terms of it, and $r, r_0, r_1, f \in \mathbb{Q}(x)$ are parameters of the three following transformations which preserve the order of the operator:

(i) change of variables $\xrightarrow{f}_C$: $y(x) \rightarrow y(f(x))$,

(ii) exp-product $\xrightarrow{r}_E$: $y \rightarrow \exp\left(\int r\,dx\right) y$, and

(iii) gauge transformation $\xrightarrow{r_0, r_1}_G$: $y \rightarrow r_0 y + r_1 y'$.

Example of irreducible second-order holonomic differential equations

Let us consider the following differential equation

$$(E3): \quad 4(x-2)^2y'' + (4x-8)y' + (-144x^4 + 1152x^3 - 3456x^2 + 4608x - 2305)y = 0.$$

By using Ruben Debeerst code, it comes out that a solution of $(E3)$ is given by

$$y(x) = \mathrm{B}_{\frac{1}{4}}\left(3(x-2)^2\right), \tag{5}$$

where $\mathrm{B}_\nu(x)$ is the Bessel function of parameter ν.

Bessel functions belong to the class of special functions since they are expressed in terms of the most prominent special function solutions of holonomic differential equations called generalized hypergeometric functions.

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Generalized hypergeometric functions

The generalized hypergeometric series ${}_pF_q$ are defined by

$$ {}_pF_q\left(\begin{array}{c} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{array} \middle| x \right) = \sum_{k=0}^{+\infty} \frac{(a_1)_k \cdot (a_2)_k \cdots (a_P)_k}{(b_1)_k \cdot (b_2)_k \cdots (b_q)_k \cdot k!} x^k , $$

where $(a)_k$ denotes the *Pochhammer symbol*

$$ (a)_k := \begin{cases} 1 & \text{if } k = 0 \text{ or } a = 0, \\ a \cdot (a+1) \cdots (a+k-1) & \text{if } k > 0 \text{ and } a \neq 0. \end{cases} \tag{6} $$

Generalized hypergeometric functions

The generalized hypergeometric series ${}_pF_q$ satisfy the holonomic differential equation

$$\delta(\delta + b_1 - 1)\cdots(\delta + b_q - 1)y(x) = x(\delta + a_1)\cdots(\delta + a_p)y(x) \qquad (7)$$

where δ denotes the differential operator $\delta = x\frac{d}{dx}$. This equation has order $\max(p, q + 1)$.

Generalized hypergeometric functions

1. For $p \leq q$ the series ${}_pF_q$ is convergent for all x,
2. for $p > q+1$ the radius of convergence is zero,
3. for $p = q+1$ the series converges for $|x| < 1$.

For $p \leq q+1$ the series and its analytic continuation is called hypergeometric function, and for $p < q+1$ the series is called generalized hypergeometric function. We are interested here by the case $p < q+1$ for which the radius of convergence is infinity.

Third-order holonomic differential equations

Third-order holonomic differential equations are of type

$$(eq): \quad a_3(x)\, y''' + a_2(x)\, y'' + a_1(x)\, y' + a_0(x)\, y = 0$$

with $a_0(x), a_1(x), a_2(x), a_3(x) \in \mathbb{Q}[x]$, $a_3(x) \neq 0$.

If **the differential operator associated to** (eq) **is reducible**, solutions of (eq) can be in some cases easily computed, since we know how to solve first-order holonomic differential equations and also, in some particular cases, second-order holonomic differential equations.

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Example of reducible third-order holonomic differential equations

Let us consider the following differential equation

$$(E4): \; 9x^6(x+7)^2y''' + 9x^4(x+7)(8x^2+47x+84)y'' + x^2(2954x^2-882x+90x^4+972x^3-21609)y' + (-74088x-259308-1274x^3-13818x^2)y = 0.$$

The operator associated to $(E4)$ can be factorized in the following form

$$(x^2D_x+12)\cdot\left(9x^4(x+7)^2D_x^2 + 9(7+2x)x^3(x+7)D_x - 21609 - 637x^2 - 6174x\right).$$

Hence, solving

$$(E5): \quad 9x^4(x+7)^2y'' + 9(7+2x)x^3(x+7)y - (21609+637x^2+6174x)y = 0$$

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By using Ruben Debeerst code, it comes out that a solution of

$$(E5): \quad 9x^4(x+7)^2y'' + 9(7+2x)x^3(x+7)y - (21609 + 637x^2 + 6174x)y = 0$$

is given in terms of Bessel functions:

$$y(x) = \mathrm{B}_{\frac{2}{3}}\left((x+7)/x\right) \tag{8}$$

which is therefore a solution of $(E4)$.

Third-order holonomic differential equations

If **the differential operator associated to**

$$(eq): \quad a_3(x)\, y''' + a_2(x)\, y'' + a_1(x)\, y' + a_0(x)\, y = 0$$

is irreducible, it is difficult to solve this equation. In addition to being irreducible, if Liouvillian or Eulerian solutions of L are not allowed, then no algorithm for this case is yet published. That is the case for example where this operator comes from certain special and useful functions such as hypergeometric functions. And that is also why we focus on those operators here.

Example of irreducible third-order holonomic differential equations

Let us consider the following third-order irreducible holonomic differential equation satisfied by the square of the Hermite polynomial:

$$(E6): \quad y''' - 6xy'' + (8x^2 + 8n - 2)y' - 16xny = 0.$$

If we assume that we don't know where $(E6)$ is coming from, it will be difficult to solve it. But, by using one of my code which I have implemented in a Maple package called `Solver3` which can be downloaded from `http://www.mathematik.uni-kassel.de/~merlin/`, we have some solutions. The Maple function is called `Hyp1F1sqSolutions` and takes as input any irreducible third-order linear differential operator L and returns, if they exist, all the parameters of transformations $(r, r_2, r_1, r_0, f \in k(x))$ and the parameters of the function ${}_1F_1^2$ that we are solving $(E6)$ in terms of it.

Start Maple

Our main objective

We develop a complete algorithm to detect the solutions of any third-order irreducible holonomic differential equation which are related to the following special functions: ${}_1F_1^2,\ {}_0F_2,\ {}_1F_2,\ {}_2F_2$.

Remark

If y a is solution of a second-order holonomic differential equation, then y^2 is a solution of a holonomic differential equation of order three. That is the case for the function $y = {}_1F_1$. This is a rich source for third-order holonomic differential equations whose solutions are sought.

All our special functions ${}_1F_1^2,\ {}_0F_2,\ {}_1F_2,\ {}_2F_2$ satisfy third-order holonomic irreducible differential equations. For example, the differential operator associated to the ${}_1F_1^2$ function is

$$\mathrm{L}_{11}^2 = x^2 D_x^3 + 3x\left(-x+b\right) D_x^2 - \left(-2x^2 + 4x(a+b) - b(2b-1)\right) D_x - 2a\left(-2x+2b-1\right)$$

where a and b are the upper and lower parameters of ${}_1F_1$, respectively.

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where a and b are the upper and lower parameters of ${}_1F_1$, respectively.

The method

To solve our differential equations, we use the three transformations

(i) change of variables $\xrightarrow{f}_C$: $y(x) \to y(f(x))$,

(ii) exp-product $\xrightarrow{r}_E$: $y \to \exp\left(\int r\,dx\right) y$, and

(iii) gauge transformation $\xrightarrow{r_0, r_1, r_2}_G$: $y \to r_0 y + r_1 y' + r_2 y''$.

where $r, r_0, r_1, f \in \mathbb{Q}(x)$.

Our goal is to find all solutions which can be written in the form:

$$\exp\left(\int r\,dx\right)\left(r_0 S(f(x)) + r_1 (S(f(x)))' + r_2 (S(f(x)))''\right)$$

where $S(x) \in \{{}_1F_1^2,\ {}_0F_2,\ {}_1F_2,\ {}_2F_2\}$ and $r, r_0, r_1, r_2, f \in \mathbb{Q}(x)$ and $r, r_0, r_1, r_2, f \in \mathbb{Q}(x)$.

Our aim

Our aim becomes now, for a given third-order irreducible holonomic differential equation, to find (if they exist) the transformation parameters r, r_0, r_1, r_2 and f, and also the parameter(s) associated to our chosen special function $S(x) \in \{{}_1F_1^2,\ {}_0F_2,\ {}_1F_2,\ {}_2F_2\}$.

Remark

The only case which is not covered since it requires different method, is where the special function has finite radius of convergence: ${}_2F_1^2$ and ${}_3F_2$ are the only examples.

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Our inputs

As input, we consider a third-order irreducible holonomic differential equation with coefficients in $\mathbb{Q}[x]$

$$(eq): \quad a_3(x)\, y''' + a_2(x)\, y'' + a_1(x)\, y' + a_0(x)\, y = 0$$

that we want to solve in term of $S(x) \in \{{}_1F_1^2,\ {}_0F_2,\ {}_1F_2,\ {}_2F_2\}$.

Let us call L the differential operator associated to (eq).
L_S is always the operator associated to the differential equation satisfied by $S(x) \in \{{}_1F_1^2,\ {}_0F_2,\ {}_1F_2,\ {}_2F_2\}$.

Steps of the resolution

1. Find the singularities of the differential operator L
2. Find the generalized exponents of L
3. Find the transformation parameter(s) which are:
   - (a) the change of variable parameter f
   - (b) the parameter(s) of our chosen special function S
   - (c) the exp-product parameter r
   - (d) the gauge parameters r_0, r_1 and r_2.

Singularities

Definition

A point $p \in \mathbb{C} \cup \{\infty\}$ is called a *singularity* of a holonomic differential operator L, if p is a zero of the leading coefficient of L. All the other points are called *regular* points.

Remark

- To understand the singularity at $x = \infty$, one can always use the change of variables $x \to \frac{1}{x}$ and deal with 0.
- At all regular points of L we can find a *fundamental system of power series solutions.*

If $p \in \mathbb{C} \cup \{\infty\}$, we define the *local parameter* t_p as

$$t_p = \begin{cases} x - p & \text{if } p \neq \infty, \\ \frac{1}{x} & \text{if } p = \infty \, . \end{cases}$$

Singularities

Let $\mathrm{L}_{\frac{1}{x}}$ denote the operator coming from L by the change of variables $x \to \frac{1}{x}$.

Definition

A singularity p of L is called

(i) *apparent singularity* if all solutions of L are regular at p,

(ii) *regular singular* ($p \neq \infty$) if $t_p^i \dfrac{a_{3-i}(x)}{a_3(x)}$ is regular at p for $1 \leq i \leq 3$,

(iii) regular singular ($p = \infty$) if $\mathrm{L}_{\frac{1}{x}}$ has a regular singularity at $x = 0$, and

(iv) *irregular singular* otherwise.

The operators coming from our functions ${}_1F_1^2$, ${}_0F_2$, ${}_1F_2$, ${}_2F_2$ have two singularities: one regular at $x = 0$ and the other irregular at $x = \infty$.

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Generalized exponents

Let us consider the following differential equation

$$(E7): \quad x^2(x+1)y'' - (6x+7x^2)y' + (12+16x)y = 0.$$

By searching a solution of $(E7)$ at the neighborhood of $x=0$ in the form $x^c \cdot G$ with $c \in \mathbb{Q}$, $G \in \mathbb{Q}[x][\ln(x)]$ such that G has a non-zero constant coefficient, we get

$$y(x) = x^3\left(1 + x - x\ln(x)\right), \tag{9}$$

and therefore, $c = 3$.

What happens if we want solutions in the same form but with c which is not a constant? The answer to this question leads us to the definition of the generalized exponent.

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Generalized exponents

Definition

Let p be a point with local parameter t_p. An element $e \in \mathbb{Q}[{t_p}^{-1/n}]$, $n \in \mathbb{N}^*$ is called a *generalized exponent* of L at the point p if there exists a *formal solution* of L of the form

$$y(x) = \exp\left(\int \frac{e}{t_p}\, dt_p\right) G, \quad G \in \mathbb{Q}(({t_p}^{1/n}))[\ln(t_p)], \tag{10}$$

where the constant term of the *Puiseux series* G is non-zero. For a given solution this representation is unique and $n \in \mathbb{N}$ is called the *ramification index* of e.

The set of generalized exponents at a point p is denoted by $\mathrm{gexp}(\mathrm{L}, p)$. Similarly, we call e a generalized exponent of the solution y at the point p if $y = y(x)$ has the representation (10) for some $G \in \mathbb{Q}(({t_p}^{1/n}))[\ln(t_p)]$. There is an algorithm called `gen-exp` given by Mark van Hoeij which computes the generalized exponents of L at a given point.

Generalized exponents

Generalized exponents and singularities

- p is an irregular singularity of L if L has at p at least one non-constant generalized exponent.
- p is a regular point of L if the three generalized exponents of L at p are $0, 1, 2$.
- If p is an apparent singular of L, then all the generalized exponents of L at p are non-negative integers.
- If p is a regular singular of L, then all the generalized exponents of L at p are constants.

Generalized exponents

Example

The operator L_{11}^{2} coming from the function ${}_1F_1^2$ with parameters a and b (the upper and lower parameters of ${}_1F_1$, respectively) has as generalized exponents:

1. at its regular singularity $x = 0$: $[0,\ 1-b,\ 2(1-b)]$,
2. at its irregular singularity $x = \infty$:

$$\left[2a,\ -2t^{-1} + 2(b-a),\ -t^{-1} + b\right] \text{ with } t = \frac{1}{x}.$$

Generalized exponents and exp-product transformation

Lemma

Let $L_1, L_2 \in \mathbb{Q}[x][\partial]$ be two irreducible third-order holonomic differential operators such that $L_1 \xrightarrow{r}_E L_2$ and let e be a generalized exponent of L_1 at the point $p \in \mathbb{C} \cup \{\infty\}$ with the ramification index $n_e \in \mathbb{N}^*$. Furthermore, let r has at p the series representation

$$r = \sum_{i=m_p}^{+\infty} r_i t_p^i, \quad m_p \in \mathbb{Z} \text{ with } r_i \in \mathbb{Q} \text{ and } r_{m_p} \neq 0.$$

where t_p is the local parameter of p.

Generalized exponents and exp-product transformation

1- If p is not a pole of r then $m_p \geq 0$ and the generalized exponent of L_2 at p is

$$\overline{e} = \begin{cases} e & \text{if } p \neq \infty, \\ e - r_0 t_\infty^{-1} - r_1 & \text{otherwise.} \end{cases} \quad (11)$$

2- If p is a pole of r then $m_p \leq -1$, where $-m_p$ is the multiplicity order of r at p, and the generalized exponent of L_2 at p is given by

$$\overline{e} = \begin{cases} e + \sum_{i=m_p}^{-1} r_i t_p^{i+1} & \text{if } p \neq \infty, \\ e - \sum_{i=m_\infty}^{1} r_i t_\infty^{i-1} & \text{otherwise.} \end{cases} \quad (12)$$

Generalized exponents and gauge transformation

Lemma

Let $\mathrm{L}_1, \mathrm{L}_2 \in \mathbb{Q}[x][\partial]$ be two irreducible third-order holonomic differential operators such that $\mathrm{L}_1 \longrightarrow_G \mathrm{L}_2$ and let e be a generalized exponent of L_1 at some point p. The operator L_2 has at p a generalized exponent $\overline{e}$ such that $\overline{e} = e \bmod \frac{1}{n_e}\mathbb{Z}$, where $n_e \in \mathbb{N}^*$ is the ramification index of e.

Generalized exponents and change of variable transformation

Let us consider the case $\mathrm{L}_S \xrightarrow{f}_C \mathrm{M}$ with $S(x) \in \{{}_1F_1^2,\ {}_0F_2,\ {}_1F_2,\ {}_2F_2\}$ and $f \in \mathbb{Q}(x) \setminus \mathbb{Q}$. Since L_S has singularities at 0 and ∞, we will see how the generalized exponents of M look like at the points p such that $f(p) = 0$ and $f(p) = \infty$ (i.e. at the *zeroes* and *poles* of f).
We have given and proved a general theorem on the behaviour of the generalized exponents of M after a change of variable at the zeroes and poles of f. Let us apply this theorem where L_S is the operator L_{11}^2 coming from the function ${}_1F_1^2$ with parameters a and b (the upper and lower parameters of ${}_1F_1$, respectively).

Generalized exponents and change of variable transformation

Generalized exponents at the zeroes of f

Let p be a zero of f with *multiplicity* $m_p \in \mathbb{N}^*$. Then the generalized exponents of M at p are

$$[0, \;\; m_p\,(1-b)\,, \;\; 2m_p\,(1-b)]\,.$$

Generalized exponents and change of variable transformation

Generalized exponents at the poles of f

Let p be a pole of f with *multiplicity* $m_p \in \mathbb{N}^*$, then the generalized exponents of M at p are

$$\left[2m_p a, \quad -2m_p(a-b)+2\sum_{j=-m_p}^{-1} j f_j t_p^j, \quad m_p b + \sum_{j=-m_p}^{-1} j f_j t_p^j\right],$$

where $f = t_p^{-m_p} \sum_{j=0}^{+\infty} f_{j-m_p} t_p^j$ with $f_{j-m_p} \in \mathbb{Q}$ and $f_{-m_p} \neq 0$.

Transformations

The task now is to find our three transformations such that

$$\mathrm{L}_S \xrightarrow{f}_C \mathrm{M} \xrightarrow{r}_E \mathrm{L}_1 \xrightarrow{r_0, r_1, r_2}_G \mathrm{L}$$

with $r, r_0, r_1, r_2 \in \mathbb{Q}(x)$, $f^2 \in \mathbb{Q}(x) \setminus \mathbb{Q}$ and $\mathrm{M}, \mathrm{L}_1 \in \mathbb{Q}[x][\partial]$.
A solution y of L in terms of S will be

$$y = \exp\left(\int r\,dx\right)\left(r_0 S(f(x)) + r_1 (S(f(x)))' + r_2 (S(f(x)))''\right).$$

Finding these transformations is equivalent to find their parameter(s). We proceed as follows:

1. in the first step, we find the change of variable parameter f and the parameter(s) associated to the function S,
2. in the second step, we find the parameters r, r_0, r_1 and r_2 for the exp-product and gauge transformations.

How to find the change of variable parameter f

Let $S \in \{{}_1F_1^2,\ {}_0F_2,\ {}_1F_2,\ {}_2F_2\}$.

1. If the ramification index of L_S at ∞ is 1 we compute the polar part of f from the generalized exponents of L at its irregular singularities. Then we get f by using the regular singularities of L or some information related to the degree of the numerator that f can have.
2. If the ramification index of L_S at ∞ is greater than 1, we put f in the form $f = \dfrac{A}{B}$ with $A, B \in k[x]$, B monic and $\gcd(A, B) = 1$. Using the generalized exponents at the irregular singularities of L, we can compute B and a bound for the degree of A. Hence, using also the fact that the ramification index of L_S is greater than 1, we can get the truncated series for f and some linear equations for the coefficients of A. By comparing the number of linear equations for the coefficients of A and the degree of A, we will deal with some cases which will help us to find A.

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How to find the parameter(s) of our chosen special function S

Let us assume that we know f. By observing the forms of the generalized exponents of L at the zeroes and poles of f with their corresponding multiplicity orders, we find the parameter(s) of our special function S. Therefore, we obtain the differential operator L_S associated to S. That means we get the operator M coming from L_S by the change of variable transformation with parameter f.

$$\mathrm{L}_S \xrightarrow{f}_C \mathrm{M}.$$

How to find the exp-product parameter r

Let us assume that we know the operator M such that

$$\mathrm{L}_S \xrightarrow{f}_C \mathrm{M} \xrightarrow{r}_E \mathrm{L}_1 \xrightarrow{r_0, r_1, r_2}_G \mathrm{L}$$

To find the exp-product parameter r we proceed as follows:

Theorem

Let $\mathrm{M}, \mathrm{L} \in k(x)[\partial]$ be two irreducible third-order linear differential operators such that $\mathrm{M} \longrightarrow_{EG} \mathrm{L}$ and r the parameter of the exp-product transformation. Let $\mathbb{NS}$ be the set of all non-apparent singularities of L and $\mathbb{P}_0$ the set of all the poles of r. For $p \in \mathbb{P}_0 \cup \mathbb{NS}$, let us set

$$e_p^i = e_p^i(\mathrm{L}) - e_p^i(\mathrm{M}), \quad i = 1, 2, 3$$

where $e_p^i(\mathrm{M})$ and $e_p^i(\mathrm{L})$ are the i-th generalized exponent of M and L at p, respectively. Then the exp-product parameter r is given by

$$r = \sum_{p \in \mathbb{NS} \setminus \{\infty\}} \frac{e_p^i}{t_p} - t_\infty \cdot \overline{e_\infty^i} + \sum_{p \in \mathbb{NS} \setminus \{\infty\}} \frac{c_p}{n_p} t_p^{-1} \qquad (13)$$

How to find the exp-product parameter r

where $\overline{e_\infty^i} = e_\infty^i - \mathrm{const}(e_\infty^i)$ with $\mathrm{const}(e_\infty^i)$ the constant term of e_∞^i, $n_p = \max\left\{n_{e_p^i(\mathrm{M})}, i = 1, 2, 3\right\}$ with $n_{e_p^i(\mathrm{M})}$ the ramification index of $e_p^i(\mathrm{M})$, and $c_p \in \mathbb{Z}$ with $|c_p| < n_p$.

How to find the gauge parameters r_0, r_1 and r_2

Let us assume that we get our exp-product parameter r. Therefore, we have the operator L_1 such that

$$\mathrm{L}_S \xrightarrow{f}_C \mathrm{M} \xrightarrow{r}_E \mathrm{L}_1 \xrightarrow{r_0, r_1, r_2}_G \mathrm{L}$$

To find the gauge parameters r_0, r_1 and r_2, if they exist, we use the gauge equivalence test (from Mark Van Hoeij) which gives us those parameters as the coefficients of a second-order linear differential operator: $r_2\partial^2 + r_1\partial + r_0$.

At the end, if we succeed to find those parameters of transformations and also the parameter(s) associated to the considered special function S, we get a solution y of L in the form

$$y(x) = \exp\left(\int r\,dx\right)\left(r_0 S(f(x)) + r_1 (S(f(x)))' + r_2 (S(f(x)))''\right).$$

How to find the gauge parameters r_0, r_1 and r_2

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$$y(x) = \exp\left(\int r\,dx\right)\left(r_0 S(f(x)) + r_1(S(f(x)))' + r_2(S(f(x)))''\right).$$

Illustration example

We have implemented the methods of this work in a Maple package called `Solver3` which can be downloaded from `http://www.mathematik.uni-kassel.de/~merlin/`. My PhD thesis explains the algorithms in more detail. We will take here just one example and show how our package can be used. Let us consider the following third-order irreducible holonomic differential equation satisfied by the square of the Laguerre polynomial:

$$x^2y'''+(-3x^2+3x)y''+(4nx+2x^2-4x+1)y'+(-4nx+2n)y=0. \quad (14)$$

Let our input operator L be the operator associated to this differential equation and see if we can solve it using our codes. That means if we can find the function
$S \in \{{}_1F_1^2,\ {}_0F_2,\ {}_1F_2,\ {}_2F_2\}$ and the transformation parameters such that

$$\mathrm{L}_S \xrightarrow{f}_C \mathrm{M} \longrightarrow_{EG} \mathrm{L}.$$

We upload first our Maple package called `Solver3`.
We show step by step in the Maple file below how we use our method to solve this example.

Illustration example

Start Maple

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Thank for your kind attention!!!