# Generating functions and hypergeometric representations of classical continuous orthogonal polynomials

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# Overview

Let $F_n$, $n \in \mathbb{N}$, be the Fibonacci numbers defined by

$$F_{n+1} = F_n + F_{n-1}, \, (n \geq 1; \, F_0 = 0; \, F_1 = 1).$$

Multiplying both sides with $t^n$ and summing from $n = 2$ to $\infty$, we obtain

$$\begin{aligned} \sum_{n=1}^{+\infty} F_{n+1} t^n &= \sum_{n=1}^{+\infty} F_n t^n + \sum_{n=1}^{+\infty} F_{n-1} t^n \\ &= \sum_{n=0}^{+\infty} F_n t^n + \sum_{n=0}^{+\infty} F_n t^{n+1} \\ &= F(t) + tF(t), \quad F(t) = \sum_{n=0}^{+\infty} F_n t^n. \end{aligned}$$

Observing that

$$\sum_{n=1}^{+\infty} F_{n+1} t^n = \sum_{n=2}^{+\infty} F_n t^{n-1} = t^{-1}\left(\sum_{n=1}^{\infty} F_n t^n - t\right),$$

we obtain

$$\frac{F(t)-t}{t} = F(t) + tF(t).$$

That is

$$F(t) = \frac{t}{1-t-t^2}.$$

The function $F$ is called a generating function of the series

$$\sum_{n=0}^{+\infty} F_n t^n$$

.

## Definition

Let $\{p_n(x)\}_{n=0}^{\infty}$ be a sequence of polynomials. A genrating function of $\{p_n(x)\}_{n=0}^{\infty}$ is the function $G(x,t)$ defined by

$$G(x,t) = \sum_{n=0}^{\infty} c_n P_n(x) t^n, \tag{1}$$

where $\{c_n\}_{n=0}^{\infty}$ is a sequence of real or complex numbers.

## Example

Find Generating function of the sequence $\{x^n\}_{n=0}^{\infty}$ in the following cases

1) $c_n = 1,\ n = 0, 1, 2, ....$ 2)$c_n = \frac{1}{n!},\ n = 0, 1, 2, ...$

## Solution

1. *Taking* $P_n(x) = x^n$ *and* $c_n = 1,\ n = 0, 1, 2, \ldots$ *into (1) we obtain*

$$\sum_{n=0}^{\infty} x^n t^n = \frac{1}{1 - xt},\ |xt| < 1.$$

2. *Taking* $P_n(x) = x^n$ *and* $c_n = \frac{1}{n!},\ n = 0, 1, 2, \ldots$ *into (1) we obtain*

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} t^n = \exp(xt).$$

Generating functions have many applications in mathematics. For instance they can be used to

1. Find an exact formula for the members of a sequence
2. Find a recurrence formula
3. Find asymptotic formula for a sequence

## Objective

In this part, we show how to obtain a generating function of a sequence of classical continuous orthogonal polynomials and derive their hypergeometric representation.

Let us recall that a sequence $\{p_n\}_n$ of continuous polynomials, orthogonal with respect to a weight function $\rho$ is classical if and only if there exists a polynomial $\phi$ of degree at most two and a sequence $\{A_n\}_n$ of numbers such that

$$p_n(x) = \frac{A_n}{\rho(x)} \frac{d^n}{dx^n}[\phi(x)^n \rho(x)]. \tag{2}$$

## Definition

Generating function of classical continuous orthogonal polynomials is the function G(x,t) for which the series expansion in a neighbourhood of $t = 0$, is

$$G(x,t) = \sum_{n=0}^{\infty} \frac{p_n(x)}{A_n n!} t^n, \tag{3}$$

where $A_n$ is the coefficient in (2).

### Theorem

*Let $\{p_n\}_n$ be a sequence of continuous orthogonal polynomials that satisfies the Rodriguess formula (2) . The function*

$$G(x,t) = \frac{\rho(z)}{\rho(x)} \frac{1}{1-\phi'(z)t}\bigg|_{z=\zeta(x,t)} \tag{4}$$

*is a generation function of $\{p_n\}_n$, where $\zeta(x,t)$ is the zero of $z - x - \phi(z)t$ satisfying $\lim\limits_{t\to 0}\zeta(x,t) = x$.*

# Proof

Since $p_n$, $n = 0, 1, 2, \ldots$, is continuous and satisfies the Rodriguess formula

$$p_n(x) = \frac{A_n}{\rho(x)} \frac{d^n}{dx^n} [\phi(x)^n \rho(x)]$$

the function given by

$$G(x, t) = \sum_{n=0}^{\infty} \frac{p_n(x)}{A_n n!} t^n$$

is a generating function of $\{p_n\}_n$. Let $x$ in the interval of orthogonality of $p_n$ and let $\mathcal{C}$ be a circle containing $x$. We obtain from the Cauchy's formula

$$\frac{d^n}{dx^n} [\phi(x)^n \rho(x)] = \frac{n!}{2i\pi} \int_{\mathcal{C}} \frac{\phi(z)^n \rho(z) dz}{(z - x)^{n+1}},$$

where $\mathcal{C}$ is a circle containing $x$.

# Proof

So, the Rodriguess formula of $p_n$ becomes

$$p_n(x) = \frac{A_n}{\rho(x)} \frac{n!}{2i\pi} \int_{\mathcal{C}} \frac{\phi(z)^n \rho(z) dz}{(z-x)^{n+1}}$$

and the generating function of $\{p_n\}$ read

$$G(x,t) = \sum_{n=0}^{\infty} \frac{1}{2i\pi\rho(x)} \int_{\mathcal{C}} \frac{(\phi(z)t)^n \rho(z) dz}{(z-x)^{n+1}}.$$

# Proof

The function $f : z \mapsto \frac{\phi(z)}{z-x}$ is bounded on the compact set $\mathcal{C}$ for is continuous there. So, for $|t| < \frac{1}{3M}$, where $M$ is an upper bound of $f$, $\left|\frac{\phi(z)t}{z-x}\right|^n < \frac{1}{3^n}$ for all $z \in \mathcal{C}$. Therefore, we can interchange the summation and integral to obtain

$$\begin{aligned} G(x,t) &= \frac{1}{2i\pi\rho(x)} \int_{\mathcal{C}} \sum_{n=0}^{\infty} \frac{(\phi(z)t)^n \rho(z)dz}{(z-x)^{n+1}} \\ &= \frac{1}{2i\pi\rho(x)} \int_{\mathcal{C}} \frac{\rho(z)dz}{z-x-\phi(z)t}. \end{aligned} \tag{5}$$

# Proof

To end the proof, we evaluate the above integral by using the Residues formula.
If $\phi$ is a polynomials of degree two, one of the zeros of the denominator $p(z) = z - x - \phi(z)t$ in the integrand tends to $\infty$ when t tends to 0 and the other one, $\zeta(x, t)$ tends to $x$ when t tends to 0.
If the polynomial $\phi$ is of degree at most one, the zero of $p(z)$ tends to $x$ when t tends to 0. So, for $|t|$ sufficiently small, the integrand in

$$G(x, t) = \frac{1}{2i\pi\rho(x)} \int_{\mathcal{C}} \frac{\rho(z)dz}{z - x - \phi(z)t}$$

has a single pole, $\zeta(x, t)$, inside the circle $\mathcal{C}$. Therefore, using the residues formula, we obtain

# Proof

$$
\begin{aligned}
G(x,t) &= \frac{1}{\rho(x)} \lim_{z\to\zeta(x,t)} \frac{(z-\zeta(x,t))\rho(z)}{z-x-\phi(z)t}, \\
&= \frac{\rho(z)}{\rho(x)} \left.\frac{1}{1-\phi'(z)t}\right|_{z=\zeta(x,t)} .
\end{aligned}
$$

# Generating function of Hermite polynomials

The Hermite polynomials $\{H_n\}_n$ are orthogonal polynomials associated with the weight $\rho(x) = \exp(-x^2)$ on the real line $\mathbb{R} = (-\infty, +\infty)$. They are known to satisfy the Rodriguess formula

$$H_n(x) = (-1)^n \exp(x^2) \frac{d^n}{dx^n} \left[\exp(-x^2)\right].$$

Identifying with the formula (2), we obtain $\phi(x) = 1$, $\rho(x) = \exp(-x^2)$ and $A_n = (-1)^n$.
Therefore $z - x - \phi(z)t = z - x - t$ has only one zero $z = x + t$.

# Generating function of Hermite polynomials

Hence, from (3) and (4), we have

$$\sum_{n=0}^{\infty} \frac{H_n(x)}{(-1)^n n!} t^n = \left. \frac{\rho(z)}{\rho(x)} \right|_{z=x+t} = \exp(-2xt - t^2).$$

Taking $-t$ for $t$, we obtain

$$G(x,t) = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n, \quad G(x,t) = \exp(2xt - t^2). \qquad (6)$$

# Generating function of Laguerre polynomials

The Laguerre polynomials $\{L_n\}_n$ are orthogonal polynomials associated with the weight $\rho(x) = x^\alpha \exp(-x)$ on the half-line $\mathbb{R}_+ = (0, +\infty)$ $(\alpha > -1)$. They are known to satisfy the Rodriguess formula

$$L_n^{(\alpha)}(x) = \frac{1}{n!} x^{-\alpha} \exp(x) \frac{d^n}{dx^n}[x^n x^\alpha \exp(-x)],$$

which is of the form (2) with $\rho(x) = x^\alpha \exp(-x)$, $\phi(x) = x$ and $A_n = \frac{1}{n!}$. Therefore the polynomial $p(z) = z - x - \phi(z)t = z - x - zt$ has only one zero $\zeta(x, t) = \frac{x}{1-t}$.

# Generating function of Laguerre polynomials

Hence, from (3) and (4)

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x)t^n = \frac{\rho(z)}{\rho(x)}\,\frac{1}{1-t}\bigg|_{z=\frac{x}{1-t}} = \frac{\rho(\frac{x}{1-t})}{(1-t)\rho(x)}.$$

Taking into account the fact that $\rho(x) = x^{\alpha}\exp(-x)$, we obtain

$$G(x,t) = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x)t^n, \quad G(x,t) = (1-t)^{-\alpha-1}\exp\left(\frac{xt}{t-1}\right). \tag{7}$$

# Generating function of Bessel polynomials

The Bessel polynomials $\{y_n(x;a)\}_n$ are orthogonal with respect to the weight function $\rho(x) = x^a \exp(-\frac{2}{x})$ on the interval $(0, +\infty)$. These polynomials are known to satisfy the Rodriguess formula

$$y_n(x;a) = 2^n x^{-a} \exp(\frac{2}{x}) \frac{d^n}{dx^n} [x^{2n+a} \exp(-\frac{2}{x})],$$

which is of he form (2) with $\rho(x) = x^a \exp(-\frac{2}{x})$, $\phi(x) = x^2$ and $A_n = 2^{-n}$. Therefore, the polynomial $p(z) = z - x - \phi(z)t = z - x - z^2 t$ has two zeros

$$\zeta_1(x,t) = \frac{1+\sqrt{1-4tx}}{2t} \text{ and } \zeta_2(x,t) = \frac{1-\sqrt{1-4tx}}{2t}.$$

# Generating function of Bessel polynomials

Since $\lim_{t\to 0} \zeta_1(x,t) = \infty$ and $\lim_{t\to 0} \zeta_2(x,t) = x$, we deduce from (3) and (4)

$$\begin{aligned} \sum_{n=0}^{\infty} 2^{-n} y_n(x;a) t^n &= \frac{\rho(z)}{\rho(x)} \frac{1}{1-2zt}\bigg|_{z=\zeta_2(x,t)}, \\ &= \frac{\rho(\zeta_2(x,t))}{\rho(x)} \frac{1}{1-2\zeta_2(x,t)t}. \end{aligned}$$

# Generating function of Bessel polynomials

Noting that $\zeta_2(x,t) = \frac{2x}{1+\sqrt{1-4tx}}$, we obtain

$$\rho(\zeta_2(x,t)) = \left(\frac{2x}{1+\sqrt{1-4tx}}\right)^a \exp\left(-\frac{1+\sqrt{1-4tx}}{x}\right)$$

and

$$\frac{1}{1-2\zeta_2(x,t)t} = \frac{1}{\sqrt{1-4tx}}.$$

Therefore,

$$\begin{aligned}\sum_{n=0}^{\infty} 2^{-n} y_n(x;a) t^n &= (1-4tx)^{-\frac{1}{2}} \left(\frac{2}{1+\sqrt{1-4tx}}\right)^a \exp\left(\frac{1-\sqrt{1-4tx}}{x}\right),\\ &= (1-4tx)^{-\frac{1}{2}} \left(\frac{2}{1+\sqrt{1-4tx}}\right)^a \exp\left(\frac{4t}{1+\sqrt{1-4tx}}\right).\end{aligned}$$

Take $\frac{t}{2}$ for $t$ to obtain

# Generating function of Bessel polynomials

$$G(x,t) = \sum_{n=0}^{\infty} y_n(x;a)t^n \tag{8}$$

with

$$G(x;t) = (1-2tx)^{-\frac{1}{2}} \left(\frac{2}{1+\sqrt{1-2tx}}\right)^a \exp\left(\frac{2t}{1+\sqrt{1-2tx}}\right).$$

# Generating function of Jacobi polynomials

The Jacobi polynomials $\{P_n^{(\alpha,\beta)}\}_n(\ \alpha > -1,\ \beta > -1)$ are orthogonal with respect to the weight function $\rho(x) = (1-x)^\alpha(1+x)^\beta$ on the interval $(-1,1)$. These polynomials are known to satisfy the Rodriguess formula

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{n!2^n}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}[(1-x)^{\alpha+n}(1+x)^{\beta+n}], \tag{9}$$

which is of the form (2), with $\rho(x) = (1-x)^\alpha(1+x)^\beta$, $\phi(x) = (1-x^2)$ and $A_n = \frac{(-1)^n}{n!2^n}$. Therefore the polynomials $p(z) = z - x - \phi(z) = z - x - (1-x^2)t$ is of degree two with zeros

$$\zeta_1(x,t) = \frac{-1-\sqrt{1+4tx+4t^2}}{2t} \text{ and } \zeta_2(x,t) = \frac{-1+\sqrt{1+4tx+4t^2}}{2t}.$$

Noting that $\lim_{t\to 0}\zeta_1(x,t) = \infty$ and $\lim_{t\to 0}\zeta_2(x,t) = x$,

# Generating function of Jacobi polynomials

we deduce from (3) and (4) the following

$$\sum_{n=0}^{\infty}(-2)^n P_n^{(\alpha,\beta)}(x)t^n = \frac{\rho(z)}{\rho(x)}\,\frac{1}{1+2zt}\bigg|_{z=\zeta_2(x,t)}, \\ = \frac{\rho(\zeta_2(x,t))}{\rho(x)}\,\frac{1}{1+2\zeta_2(x,t)t}.$$

Since $\zeta_2(x,t)=\frac{-1+\sqrt{1+4tx+4t^2}}{2t}$, we obtain after simplification

$$\rho(\zeta_2(x,t)) = \frac{2^{\alpha+\beta}(1-x)^{\alpha}(1+x)^{\beta}}{\left(1+2t+\sqrt{1+4tx+4t^2}\right)^{\alpha}\left(1-2t+\sqrt{1+4tx4+4t^2}\right)^{\beta}},$$

$$\frac{1}{1+\zeta_2(x,t)} = \frac{1}{\sqrt{1+4tx+4t^2}}.$$

# Generating function of Jacobi polynomials

Therefore

$$\sum_{n=0}^{\infty}(-2)^n P_n^{(\alpha,\beta)}(x)t^n$$

$$= \frac{2^{\alpha+\beta}}{\left(1+2t+\sqrt{1+4tx+t^2}\right)^{\alpha}\left(1-2t+\sqrt{1+4tx+t^2}\right)^{\beta}\sqrt{1+4tx+t^2}}.$$

Taking $-\frac{t}{2}$ for $t$, we obtain

$$G(x,t) = \sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)t^n \tag{10}$$

with

$$G(x,t) = \frac{2^{\alpha+\beta}}{(1-t+R)^{\alpha}(1+t+R)^{\beta}R}, \quad R = \sqrt{1-2tx+t^2}.$$

Hypergeometric function ${}_pF_q$ is define by the series

$$ {}_pF_q\left(\begin{array}{c} a_1, ..., a_{s+1} \\ b_1, ..., b_s \end{array} ; z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k ... (a_p)_k}{(b_1)_k ... (b_q)_k} \frac{z^k}{k!}, $$

where $(a)_n$ is the factorial function defined as follow

$$ (a)_n = a(a+1)(a+2)...(a+n-1), \quad n \geq 1 $$

and

$$ (a)_0 = 1. $$

The parameters must be such that the denominator factors in the terms of the series are never zero.

When one of the numerator parameters, let us say $a_1$, equals $-n$, where $n$ is a nonnegative integer, this hypergeometric function is a polynomial in $z$.

$$ {}_pF_q\left(\begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array}; z\right) = \sum_{k=0}^{n} \frac{(-n)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k} \frac{z^k}{k!}, $$

Otherwise the radius of convergence $R$ of the hypergeometric series is given by

$$ R = \begin{cases} \infty & \text{if} \quad p < q+1 \\ 1 & \text{if} \quad p = q+1 \\ 0 & \text{if} \quad p > q+1. \end{cases} $$

We are going to use Generating functions or Sturm-Liouville equation to derive hypergeometric representations of Hermite, Laguerre, Bessel and Jacobi polynomials.

# Hypergeometric representation of Hermite polynomials

Generating function of Hermite polynomials is

$$\exp(2xt - t^2) = \exp(2xt)\exp(-t^2) = \sum_{n=0}^{\infty} \frac{(2xt)^n}{n!} \sum_{k=0}^{\infty} \frac{(-t^2)^k}{k!}$$

By means of the Rainville relation

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n}{2}]} A(k,n-2k), \tag{11}$$

where $[\frac{n}{2}]$ denotes the greatest positive integer less than or equal to $\frac{n}{2}$, we obtain

# Hypergeometric representation of Hermite polynomials

$$\exp(2xt - t^2) = \sum_{n=0}^{\infty} \sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k (2x)^{n-2k}}{(n-2k)!k!} t^n.$$

Comparing coefficients of $t^n$ in this result and in

$$\exp(2xt - t^2) = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n, \tag{12}$$

we obtain

$$H_n(x) = n! \sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k (2x)^{n-2k}}{(n-2k)!k!}.$$

# Hypergeometric representation of Hermite polynomials

Using the relations

$$(n-k)! = \frac{n!}{(-1)^k(-n)_k} \text{ and } (a)_{2n} = 2^{2n}\left(\frac{a}{2}\right)_n\left(\frac{a+1}{2}\right)_n,$$

(with $a=-n$), we obtain after simplification

$$\begin{aligned} H_n(x) &= (2x)^n \sum_{k=0}^{[\frac{n}{2}]} \left(-\frac{n}{2}\right)_k \left(-\frac{n-1}{2}\right)_n \frac{\left(-\frac{1}{x^2}\right)^k}{k!}, \\ &= (2x)^n {}_2F_0\left(\begin{matrix} -\frac{n}{2}, -\frac{n-1}{2} \\ - \end{matrix} ; -\frac{1}{x^2}\right). \end{aligned}$$

# Hypergeometric representation of Laguerre polynomials

Factorial function is an extension of the ordinary factorial since $(1)_n = n!$. It is particularly convenient to use the factorial function in the binomial expansion

$$\begin{aligned}(1-t)^{-a} &= \sum_{n=0}^{\infty} \frac{(-a)(-a-1)...(-a-n+1)}{n!}(-t)^n, \\ &= \sum_{n=0}^{\infty} \frac{(a)_n}{n!} t^n. \end{aligned} \tag{13}$$

The generating function $(1-t)^{-\alpha-1} \exp\left(\frac{xt}{t-1}\right)$ of the Laguerre polynomials can be written as follow

$$(1-t)^{-\alpha-1} \exp\left(\frac{xt}{t-1}\right) = \sum_{k=0}^{\infty} \frac{(-xt)^k}{k!}(1-t)^{-1-k-\alpha}.$$

# Hypergeometric representation of Laguerre polynomials

By means of the binomial expansion (13),

$$(1-t)^{-\alpha-1}\exp\left(\frac{xt}{t-1}\right)=\sum_{k=0}^{\infty}\frac{(-xt)^k}{k!}\sum_{n=0}^{\infty}\frac{(1+k+\alpha)_n}{n!}t^n.$$

Using the Rainville formula

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}A(k,n-k), \qquad (14)$$

we obtain

$$(1-t)^{-\alpha-1}\exp\left(\frac{xt}{t-1}\right)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-x)^k(1+k+\alpha)_{n-k}}{k!(n-k)!}t^n.$$

Comparing coefficients of $t^n$ in this result and in (7), we obtain

# Hypergeometric representation of Laguerre polynomials

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \frac{(-x)^k (1+k+\alpha)_{n-k}}{k!(n-k)!}.$$

Noting that $(a)_n = (a+k)_{n-k}(a)_k$ and using the relation $(n-k)! = \frac{n!}{(-1)^k(-n)_k}$, we obtain

$$L_n^{(\alpha)}(x) = \frac{(\alpha+1)_n}{n!} \sum_{k=0}^{n} \frac{(-n)_k}{(\alpha+1)_k} \frac{x^k}{k!} = \frac{(\alpha+1)_n}{n!} {}_1F_1\left( \begin{matrix} -n \\ \alpha+1 \end{matrix} ; x \right).$$

## Hypergeometric representation of Bessel polynomials

Use the power series expansion of $\exp(z)$ in a neighbourhood of $z=0$ to transform the generating function

$$G(x;t)=(1-2tx)^{-\frac{1}{2}}\left(\frac{2}{1+\sqrt{1-2tx}}\right)^{a}\exp\left(\frac{2t}{1+\sqrt{1-2tx}}\right)$$

of the Bessel polynomials into

$$G(x,t)=\sum_{n=0}^{\infty}\frac{t^{n}}{n!}(1-2tx)^{-\frac{1}{2}}\left(\frac{2}{1+\sqrt{1-2tx}}\right)^{a+n}.$$

Use the Rainville identity

$$(1-z)^{-\frac{1}{2}}\left(\frac{2}{1+\sqrt{1-z}}\right)^{a}={}_2F_1\left(\begin{array}{c}\frac{a+1}{2},\frac{a+2}{2}\\ a+1\end{array};z\right)$$

with $a+n$ taken for $a$ and $z=2xt$, to obtain

# Hypergeometric representation of Bessel polynomials

$$G(x,t)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{\left(\frac{a+n+1}{2}\right)_k\left(\frac{a+n+2}{2}\right)_k}{(a+n+1)_k}\frac{(2xt)^k}{k!}\frac{t^n}{n!}.$$

Take into account the relation (14) to obtain

$$G(x,t)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{\left(\frac{a+n-k+1}{2}\right)_k\left(\frac{a+n-k+2}{2}\right)_k}{(a+n-k+1)_k}\frac{(2x)^k}{(n-k)!k!}t^n.$$

Compare coefficients of $t^n$ in this result and in (8) to obtain

$$y_n(x;a)=\sum_{k=0}^{n}\frac{\left(\frac{a+n-k+1}{2}\right)_k\left(\frac{a+n-k+2}{2}\right)_k}{(a+n-k+1)_k}\frac{(2x)^k}{(n-k)!k!}.$$

# Hypergeometric representation of Bessel polynomials

Use the relations (13) and
$(a+n-k+1)_{2k} = (a+n-1+1)_k(a+n+1)_k$
to obtain

$$y_n(x;a) = \sum_{k=0}^{n} (-n)_k(a+n+1)_k \frac{\left(-\frac{x}{2}\right)^k}{k!} = {}_2F_0\left(\begin{matrix} -n, a+n+1 \\ - \end{matrix} ; -\frac{x}{2}\right).$$

# Hypergeometric representation of Jacobi polynomials

We obtain series expansion of Jacobi polynomials by means of their Sturm-Liouville equation. Jacobi polynomials are known to satisfy the Sturm-Liouville equation

$$(1-x^2)y''(x)+[\beta-\alpha-(\alpha+\beta+2)x]y'(x)+n(n+\alpha+\beta+1)y(x)=0.$$

Take

$$y(x)=\sum_{k=0}^{\infty} a_k(1-x)^k$$

and use the fact that

$$y'(x)=\sum_{k=1}^{\infty} -ka_k(1-x)^{k-1},\ y''(x)=\sum_{k=2}^{\infty} -k(k-1)a_k(1-x)^{k-2},$$

$$1-x^2=2(1-x)-(1-x)^2,\ x=1-(1-x)$$

to obtain

$$\sum_{k=0}^{\infty}[A_k a_k + B_k a_{k+1}](1-x)^k = 0,$$

where

$$\begin{aligned} B_k &= k\alpha + k\beta + k + k^2 - n^2 - n\alpha - n\beta - n, \\ C_k &= \left(-2\,\alpha - 4\,k - 2 - 2\,k\alpha - 2\,k^2\right). \end{aligned}$$

Identify coefficients of $(1-x)^k, k = 0, 1, ...$ to obtain the recurrence relation

$$a_{k+1} = \frac{(-n+k)(\alpha+\beta+1+n+k)}{2(\alpha+1+k)(k+1)} a_k.$$

Iterate this relation and take $k+1$ for $k$ to have

$$a_k = \frac{(-n)_k(\alpha+\beta+n+1)_k}{2^k(\alpha+1)_k k!} a_0.$$

Note that $(-n)_k = 0,\ k > n$, to get

# Hypergeometric representation of Jacobi polynomials

$$y(x) = a_0 \sum_{k=0}^{n} \frac{(-n)_k(\alpha+\beta+n+1)_k}{(\alpha+1)_k k!} \left(\frac{1-x}{2}\right)^k.$$

Therefore there is a constant $C$ not depending on $x$ such that

$$P_n^{(\alpha,\,\beta)}(x) = C \sum_{k=0}^{n} \frac{(-n)_k(\alpha+\beta+n+1)_k}{(\alpha+1)_k k!} \left(\frac{1-x}{2}\right)^k.$$

$$P_n^{(\alpha,\beta)}(1) = C.$$

By means of Leibnizs rule, the Rodriuess formula (9) of Jacobi polynomials read

# Hypergeometric representation of Jacobi polynomials

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n(\alpha+1)_n(\beta+1)_n}{n!2^n}\sum_{k=0}^{n}\frac{(-1)^k(1-x)^{n-k}(1+x)^k}{(\alpha+1)_{n-k}(\beta+1)_k(n-k)!k!}.$$

Hence $P_n^{(\alpha,\beta)}(1) = \frac{(\alpha+1)_n}{n!}$ and we obtain

$$\begin{aligned} P_n^{(\alpha,\beta)}(x) &= \frac{(\alpha+1)_n}{n!}\sum_{k=0}^{n}\frac{(-n)_k(\alpha+\beta+n+1)_k}{(\alpha+1)_k k!}\left(\frac{1-x}{2}\right)^k, \\ &= \frac{(\alpha+1)_n}{n!}\,{}_2F_1\left(\begin{matrix}-n,\alpha+\beta+n+1\\ \alpha+1\end{matrix};\frac{1-x}{2}\right). \end{aligned}$$

THANK YOU

- M.E.H. Ismail, and W. Van Assche: Classical and Quantum Orthogonal Polynomial in One Variable. Cambridge University Press, (2005).
- McBride Elna B.: Obtaining generating functions. Vol. 21. Springer Science and Business Media, (2012).
- Beals Richard, and Roderick Wong: Special functions and orthogonal polynomials. Vol. 153. Cambridge University Press, (2016).
- Nikiforov A.F., and Uvarov V.B.: Special functions of mathematical physics. Vol. 205. Basel: Birkhuser (1988).
- Rainville Earl D.:Special functions, (1971).