Inversion, multiplication and connection formulae for classical continuous orthogonal polynomials

D.D. TCHEUTIA

UNI KASSEL VERSITÄT

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Inversion formula of the Laguerre polynomials

We consider the Laguerre polynomials

$$L_n^{(\alpha)}(x) = \frac{(\alpha+1)_n}{n!}\,{}_1F_1\left(\begin{array}{c} -n \\ \alpha+1 \end{array}\middle| x\right) = \frac{(\alpha+1)_n}{n!}\sum_{k=0}^{n}\frac{(-n)_k}{(\alpha+1)_k}\frac{x^k}{k!}.$$

From this definition,

$$L_0^{(\alpha)}(x) = 1 \Rightarrow x^0 = 1 = L_0^{(\alpha)}(x);$$

$$L_1^{(\alpha)}(x) = -x+\alpha+1 \Rightarrow x = -L_1^{(\alpha)}(x) + (\alpha+1)L_0^{(\alpha)}(x);$$

and from

$$L_2^{(\alpha)}(x) = \frac{1}{2}x^2 - (\alpha+2)x + \frac{1}{2}(\alpha+1)(\alpha+2),$$

we get

$$x^2 = 2L_2^{(\alpha)}(x) - (2\alpha+4)L_1^{(\alpha)}(x) + (\alpha+1)(\alpha+2)L_0^{(\alpha)}(x).$$

Inversion formula

In general, since $\deg L_n^{(\alpha)}(x) = n$, the system $\{L_k^{(\alpha)}(x), k = 0, 1, 2, \dots, n\}$ is a basis of polynomials of degree at most n. Therefore, there exist coefficients $I_k(n)$ such that the inversion formula

$$x^n = \sum_{k=0}^{n} I_k(n) L_k^{(\alpha)}(x)$$

is valid. How can we compute the inversion coefficients $I_k(n)$?
One option is by using a generating function of the given polynomials.
A generating function of the Laguerre polynomials is given by

$$e^t {}_0F_1\left(\begin{array}{c} - \\ \alpha+1 \end{array} \middle| -xt \right) = \sum_{n=0}^{\infty} \frac{L_n^{(\alpha)}(x)}{(\alpha+1)_n} t^n,$$

which is equivalent to

$${}_0F_1\left(\begin{array}{c} - \\ \alpha+1 \end{array} \middle| -xt \right) = e^{-t} \sum_{n=0}^{\infty} \frac{L_n^{(\alpha)}(x)}{(\alpha+1)_n} t^n.$$

Inversion formula of the Laguerre polynomials

Using the series expansion of the exponential function, this yields

$$\sum_{n=0}^{\infty}\frac{(-xt)^n}{(\alpha+1)_n n!}=\left(\sum_{n=0}^{\infty}\frac{(-t)^n}{n!}\right)\left(\sum_{k=0}^{\infty}\frac{L_k^{(\alpha)}(x)}{(\alpha+1)_k}t^k\right).$$

From the relation (Cauchy product)

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}A(k,n-k),$$

we get

$$\sum_{n=0}^{\infty}\frac{(-1)^n x^n t^n}{(\alpha+1)_n n!}=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^{n-k}t^{n-k}}{(n-k)!}\frac{L_k^{(\alpha)}(x)}{(\alpha+1)_k}t^k$$

$$=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{(-1)^{n-k}L_k^{(\alpha)}(x)}{(n-k)!(\alpha+1)_k}\right)t^n.$$

Exercise

Equating the coefficients of t^n yields

$$x^n = \sum_{k=0}^{n} \frac{(-1)^k n!(\alpha+1)_n}{(n-k)!(\alpha+1)_k} L_k^{(\alpha)}(x).$$

Exercise: Use the generating function

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n,$$

of the Hermite polynomials and the relation

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/2 \rfloor} A(k,n-2k),$$

where $\lfloor n/2 \rfloor$ is the floor of $n/2$, to show the inversion formula

$$x^n = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n! H_{n-2k}(x)}{2^n k!(n-2k)!}.$$

Multiplication formula of the Laguerre polynomials

We consider the following generating function of the Laguerre polynomials

$$G(x,t) := e^t {}_0F_1\left(\begin{matrix} - \\ \alpha+1 \end{matrix}\middle| -xt\right) = \sum_{n=0}^{\infty} \frac{L_n^{(\alpha)}(x)}{(\alpha+1)_n} t^n.$$

From this definition, we get $G(ax,t) = e^{t(1-a)}G(x,at)$ or equivalently

$$\sum_{n=0}^{\infty} \frac{L_n^{(\alpha)}(ax)}{(\alpha+1)_n} t^n = \left(\sum_{n=0}^{\infty} \frac{(1-a)^n t^n}{n!}\right)\left(\sum_{k=0}^{\infty} \frac{L_k^{(\alpha)}(x)}{(\alpha+1)_k} a^k t^k\right).$$

It follows from the Cauchy product that

$$\sum_{n=0}^{\infty} \frac{L_n^{(\alpha)}(ax)}{(\alpha+1)_n} t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{a^k(1-a)^{n-k} L_k^{(\alpha)}(x)}{(\alpha+1)_k (n-k)!} t^n.$$

Equating the coefficients of t^n yields the multiplication formula of the Laguerre polynomials

$$L_n^{(\alpha)}(ax) = \sum_{k=0}^{n} \frac{a^k(\alpha+1)_n(1-a)^{n-k}}{(\alpha+1)_k (n-k)!} L_k^{(\alpha)}(x).$$

Exercise

Show that for the Hermite polynomials, the generating function $G(t,x) = \exp(2xt - t^2)$ satisfies

$$G(t, ax) = e^{(a^2-1)t^2} G(at, x).$$

Deduce from

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n$$

that the multiplication formula

$$H_n(ax) = \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{a^n n! (1-a^{-2})^m}{(n-2m)!m!} H_{n-2m}(x)$$

is valid.

Connection formula of the Laguerre polynomials

If we rather consider the generating function

$$G(\alpha,t) := (1-t)^{-\alpha-1}\exp\left(\frac{xt}{t-1}\right) = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x)t^n$$

of the Laguerre polynomials, we have the relation $G(\alpha,t) = (1-t)^{\alpha-\beta}G(\beta,t)$. This is equivalent to

$$\begin{aligned}\sum_{n=0}^{\infty} L_n^{(\alpha)}(x)t^n &= \left(\sum_{n=0}^{\infty}\frac{(\alpha-\beta)_n}{n!}t^n\right)\left(\sum_{k=0}^{\infty} L_k^{(\beta)}(x)t^k\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\frac{(\alpha-\beta)_{n-m}}{(n-m)!}L_m^{(\beta)}(x)\right)t^n.\end{aligned}$$

Equating the coefficients of t^n, we deduce the connection formula

$$L_n^{(\alpha)}(x) = \sum_{m=0}^{n}\frac{(\alpha-\beta)_{n-m}}{(n-m)!}L_m^{(\beta)}(x),$$

of the Laguerre polynomials.

An application of the connection formula

One application of the latter formula is the so-called parameter derivative of $L_n^{(\alpha)}(x)$ given by

$$\frac{\partial}{\partial\alpha}L_n^{(\alpha)}(x)=\sum_{m=0}^{n-1}\frac{1}{n-m}L_m^{(\alpha)}(x).$$

To get this result knowing the connection relation

$$L_n^{(\alpha)}(x)=\sum_{m=0}^{n}C_m(n;\alpha,\beta)L_m^{(\beta)}(x),$$

we build the difference quotient

$$\begin{aligned}\frac{L_n^{(\alpha)}(x)-L_n^{(\beta)}(x)}{\alpha-\beta} &= \sum_{m=0}^{n}\frac{C_m(n;\alpha,\beta)}{\alpha-\beta}L_m^{(\beta)}(x)-\frac{L_n^{(\beta)}(x)}{\alpha-\beta}\\ &= \frac{C_n(n;\alpha,\beta)-1}{\alpha-\beta}L_n^{(\beta)}(x)+\sum_{m=0}^{n-1}\frac{C_m(n;\alpha,\beta)}{\alpha-\beta}L_m^{(\beta)}(x),\end{aligned}$$

An application of the connection formula

so that with $\beta \to \alpha$

$$\frac{\partial}{\partial\alpha} L_n^{(\alpha)}(x) = \lim_{\beta\to\alpha} \frac{C_n(n;\alpha,\beta)-1}{\alpha-\beta} L_n^{(\beta)}(x) + \sum_{m=0}^{n-1} \lim_{\beta\to\alpha} \frac{C_m(n;\alpha,\beta)}{\alpha-\beta} L_m^{(\beta)}(x)$$

since the systems $L_n^{(\alpha)}(x)$ are continuous with respect to α.

$$\begin{aligned}
\lim_{\beta\to\alpha} \frac{C_m(n;\alpha,\beta)}{\alpha-\beta} &= \lim_{\beta\to\alpha} \frac{(\alpha-\beta)_{n-m}}{(\alpha-\beta)(n-m)!} \\
&= \lim_{\beta\to\alpha} \frac{(\alpha-\beta)(\alpha-\beta+1)\cdots(\alpha-\beta+n-m-1)}{(\alpha-\beta)(n-m)!} \\
&= \frac{1}{n-m},
\end{aligned}$$

and the result follows.

The data corresponding to each family

Every orthogonal polynomial sequence $\{p_n(x) = k_n x^n + \dots\}_{n\geq 0}$ is solution of a second-order differential equation of type

$$\sigma(x)y''(x) + \tau(x)y'(x) + \lambda_n y(x) = 0,$$

where $\sigma(x) = ax^2 + bx + c$, $\tau(x) = dx + e$, $d \neq 0$, $\lambda_n = -n((n-1)a + d)$. The data corresponding to each classical continuous family are given in the following table:

	Jacobi	Laguerre	Hermite	Bessel
system	$P_n^{(\alpha,\beta)}(x)$	$L_n^{(\alpha)}(x)$	$H_n(x)$	$y_n(x;\alpha)$
$\sigma(x)$	$1-x^2$	x	1	x^2
$\tau(x)$	$\beta-\alpha-(\alpha+\beta+2)x$	$\alpha+1-x$	$-2x$	$2+(\alpha+2)x$
k_n	$\frac{(\alpha+\beta+n+1)_n}{2^n n!}$	$\frac{(-1)^n}{n!}$	2^n	$\frac{(n+\alpha+1)_n}{2^n}$

The three-term recurrence relation

Furthermore, a three-term recurrence relation of type

$$xp_n(x) = a_n p_{n+1}(x) + b_n p_n(x) + c_n p_{n-1}(x),$$

is satisfied by $p_n(x)$, with

$$a_n = \frac{k_n}{k_{n+1}},$$

$$b_n = -\frac{2bn(an+d-a) - e(2a-d)}{(d+2an)(d-2a+2an)},$$

$$c_n = -\Big(n(an+d-2a)(4ac-b^2) + 4a^2c - ab^2 + ae^2 - 4acd + db^2 - bed + d^2c\Big) \times \frac{(an+d-2a)n}{(d-2a+2an)^2(2an-3a+d)(2an-a+d)} \frac{k_n}{k_{n-1}}.$$

Maple

Three-term recurrence relation for the derivatives

Since the derivatives $\{p_n'(x)\}_{n\geq 1}$ of $\{p_n(x)\}_{n\geq 0}$ is also an orthogonal polynomial sequence, it also satisfies a three-term recurrence relation of type

$$xp_n'(x) = \alpha_n p_{n+1}'(x) + \beta_n p_n'(x) + \gamma_n p_{n-1}'(x),$$

with

$$\alpha_n = \frac{n}{n+1}\frac{k_n}{k_{n+1}},$$

$$\beta_n = \frac{-2bn(an+d-a)+d(b-e)}{(d+2an)(d-2a+2an)},$$

$$\gamma_n = -\frac{n((n-1)(an+d-a)(4ac-b^2)+ae^2+d^2c-bed)(an+d-a)}{(d-2a+2an)^2(2an-3a+d)(2an-a+d)}\,\frac{k}{k_{n-}}$$

Maple

The idea

We set $x^n = v_n(x)$. Therefore, we have

$$xv_n(x) = v_{n+1}(x),\ xv_n'(x) = \frac{n}{n+1} v_{n+1}'(x).$$

We suppose that

$$x^n = v_n(x) = \sum_{m=0}^{n} I_m(n) p_m(x),$$

which means that the coefficients $I_m(n) = 0$ for $m \neq 0, 1, \ldots, n$. The idea is to find a recurrence equation satisfied by the inversion coefficients $I_m(n)$ and solve the obtained recurrence equation using the Petkovšek-van-Hoeij algorithm to get its hypergeometric term solutions.

Step 1

We substitute $v_n(x) = \sum_{m=0}^{n} I_m(n) p_m(x)$, in $x v_n(x) = v_{n+1}(x)$ to get

$$\sum_{m=0}^{n} I_m(n) x p_m(x) = \sum_{m=0}^{n+1} I_m(n+1) p_m(x).$$

Using the three-term recurrence relation

$$x p_n(x) = a_n p_{n+1}(x) + b_n p_n(x) + c_n p_{n-1}(x),$$

it follows that

$$\sum_{m=0}^{n} I_m(n) \Big(a_m p_{m+1}(x) + b_m p_m(x) + c_m p_{m-1}(x) \Big) = \sum_{m=0}^{n+1} I_m(n+1) p_m(x).$$

After a shift of index we get

$$\sum_{m=0}^{n+1} \Big(a_{m-1} I_{m-1}(n) + b_m I_m(n) + c_{m+1} I_{m+1}(n) \Big) p_m(x) = \sum_{m=0}^{n+1} I_m(n+1) p_m(x).$$

Step 2

Equating the coefficients of $p_m(x)$ yields a mixed recurrence relation with respect to m and n

$$a_{m-1}I_{m-1}(n) + b_m I_m(n) + c_{m+1}I_{m+1}(n) = I_m(n+1).$$

Similarly, we substitute

$$v_n(x) = \sum_{m=0}^{n} I_m(n)p_m(x),$$

in $xv_n'(x) = \frac{n}{n+1}v_{n+1}'(x)$ and use

$$xp_n'(x) = \alpha_n p_{n+1}'(x) + \beta_n p_n'(x) + \gamma_n p_{n-1}'(x)$$

to get, after a shift of index,

$$\sum_{m=0}^{n+1}\Big(\alpha_{m-1}I_{m-1}(n)+\beta_m I_m(n)+\gamma_{m+1}I_{m+1}(n)\Big)p_m'(x) = \frac{n}{n+1}\sum_{m=0}^{n+1} I_m(n+1)p_m'(x)$$

Recurrence equation satisfied by $I_m(n)$

By equating the coefficients of $p_m'(x)$, we get a mixed recurrence relation in the variables m and n

$$\alpha_{m-1} I_{m-1}(n) + \beta_m I_m(n) + \gamma_{m+1} I_{m+1}(n) = \frac{n}{n+1} I_m(n+1).$$

Combining the latter recurrence equation and

$$a_{m-1} I_{m-1}(n) + b_m I_m(n) + c_{m+1} I_{m+1}(n) = I_m(n+1),$$

we get out with a recurrence equation with respect to m

$$\alpha_{m-1} I_{m-1}(n)+\beta_m I_m(n)+\gamma_{m+1} I_{m+1}(n) = \frac{n}{n+1}\Big(a_{m-1} I_{m-1}(n)+b_m I_m(n)+c_{m+1} I_{m+1}(n)\Big),$$

that is

$$\left(\alpha_{m-1}-\frac{n}{n+1}a_{m-1}\right) I_{m-1}(n)+\left(\beta_m-\frac{n}{n+1}b_m\right) I_m(n)+\left(\gamma_{m+1}-\frac{n}{n+1}c_{m+1}\right) I_{m+1}(n) = 0$$

Exercise

We consider the Jacobi polynomials bases $v_n(x) = (x+1)^n$.

1. Show that

$$xv_n(x) = v_{n+1}(x) - v_n(x), \; xv_n'(x) = \frac{n}{n+1} v_{n+1}'(x) - v_n'(x).$$

2. We suppose that

$$(x+1)^n = v_n(x) = \sum_{m=0}^{n} I_m(n) p_m(x).$$

Show that $I_m(n)$ is solution of the recurrence relation

$$\left(\alpha_{m-1} - \frac{n}{n+1} a_{m-1}\right) I_{m-1}(n) + \left(\beta_m - \frac{n}{n+1}(b_m+1)+1\right) I_m(n) + \left(\gamma_{m+1} - \frac{n}{n+1} c_{m+1}\right) I_{m+1}(n) = 0.$$

Step 3

The next step is to substitute a, b, c, d, e for each family and solve the recurrence relation to get the inversion coefficients. To solve the obtained recurrence equations, we can use the Petkovšek-van-Hoeij algorithm implemented in Maple by the command

```
`LREtools/hypergeomsols`(rec,R(m),{},output=basis).
```

The solution is given up to a multiplicative constant. Let us consider the case of the Bessel polynomials.
The coefficients $I_m(n)$ of the inversion formula

$$x^n = \sum_{m=0}^{n} I_m(n) y_m(x;\alpha)$$

of the Bessel polynomials are solutions of the recurrence equation

$$2\,(m+\alpha)\,(2\,m+3+\alpha)\,(2+2\,m+\alpha)\,(1+m+\alpha)\,(m-n-1)\,I_{m-1}(n) \\ -2\,m\,(2\,m+3+\alpha)\,(2\,m-1+\alpha)\,(1+m+\alpha)\,(\alpha+2+2\,n)\,I_m(n) \\ -2\,m\,(m+1)\,(2\,m-1+\alpha)\,(\alpha+2\,m)\,(\alpha+m+n+2)\,I_{m+1}(n) = 0.$$

The Bessel polynomials

Using the Petkovšek-van-Hoeij algorithm implemented in Maple, we get (up to a multiplicative constant)

$$I_m(n) = \frac{\Gamma(m-n)\Gamma(1+m+\alpha)(\alpha/2+m+1/2)}{\Gamma(m+1)\Gamma(\alpha+m+n+2)}.$$

This means that for the Bessel polynomials

$$x^n = \sum_{m=0}^{n} \frac{\Gamma(m-n)\Gamma(1+m+\alpha)(\alpha/2+m+1/2)}{\Gamma(m+1)\Gamma(\alpha+m+n+2)} \times constant \times y_m(x;\alpha).$$

To get the constant, we equate the coefficients of x^n in both sides of the latter equation (noting that $y_n(x;\alpha) = \frac{(n+\alpha+1)_n}{2^n} x^n + \ldots$ and $\Gamma(m-n) = (-n)_m \Gamma(-n)$) to get

$$constant = \frac{2(-2)^n}{\Gamma(-n)}.$$

This leads to the inversion formula

$$x^n = (-2)^n \sum^{n} (2m+\alpha+1) \frac{(-n)_m \Gamma(\alpha+m+1)}{m!\Gamma(n+m+\alpha+2)} y_m(x;\alpha).$$

The Connection formula

In general, to find the coefficients $C_m(n)$ in the connection formula

$$p_n(x) = \sum_{m=0}^{n} C_m(n) q_m(x),$$

we combine

$$p_n(x) = \sum_{j=0}^{n} A_j(n) x^j \text{ and } x^j = \sum_{m=0}^{j} I_m(j) q_m(x),$$

which yields the representation

$$p_n(x) = \sum_{j=0}^{n} \sum_{m=0}^{j} A_j(n) I_m(j) q_m(x),$$

and then, interchanging the order of summation gives

$$C_m(n) = \sum_{j=0}^{n-m} A_{j+m}(n) I_m(j+m),$$

The Connection formula

or

$$p_n(x) = \sum_{m=0}^{n} C_{n-m}(n) q_{n-m}(x).$$

For orthogonal polynomials with even weight such as the Hermite and Gegenbauer polynomials, we have the relations

$$p_n(x) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} A_j(n) x^{n-2j} \text{ and } x^j = \sum_{m=0}^{\lfloor \frac{j}{2} \rfloor} I_m(j) q_{j-2m}(x),$$

from which we deduce

$$x^{n-2j} = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor - j} I_m(n-2j) q_{n-2j-2m}(x).$$

Finally, we combine the above two expressions and substitute m by $m-j$ to get

$$C_m(n) = \sum_{j=0}^{m} A_j(n) I_{m-j}(n-2j),$$

The Zeilberger algorithm

Since the summand $F(j,m,n) := A_j(n)I_m(j)$ of $C_m(n)$ turns out to be a hypergeometric term with respect to (j,m,n), i.e., the term ratios $F(j+1,m,n)/F(j,m,n)$, $F(j,m+1,n)/F(j,m,n)$, and $F(j,m,n+1)/F(j,m,n)$ are rational functions, Zeilberger's (combined with the Petkovšek-van-Hoeij) algorithm applies. If a hypergeometric term solution exists, the representation of $C_m(n)$ follows then from the initial values $C_n(n) = k_n/\overline{k}_n$, $C_{n+s}(n) = 0$, $s = 1,\ 2,\ \ldots$, where k_n, $\overline{k}_n$ are, respectively, the leading coefficients of $p_n(x)$ and $q_n(x)$.
For the Bessel polynomials for example,

$$A_j(n) = \frac{(-n)_j(\alpha+n+1)_j}{j!}\left(-\frac{x}{2}\right)^j,$$

and

$$I_m(j) = (-2)^j(2m+\beta+1)\frac{(-j)_m\Gamma(\beta+m+1)}{m!\Gamma(j+m+\beta+2)}.$$

Connection formula of the Bessel polynomials

It follows from Zeilberger's algorithm that the coefficients $C_m(n)$ of the connection formula

$$y_n(x;\alpha) = \sum_{m=0}^{n-m} C_m(n) y_m(x;\beta),$$

are solutions of the first-order recurrence equation

$$-(2\,m+\beta+1)\,(m+1)\,(m+\beta+2+n)\,(\alpha+n-1-m-\beta)\,C_{m+1}(n)$$
$$+(2\,m+3+\beta)\,(\beta+m+1)\,(m-n)\,(1+m+\alpha+n)\,C_m(n) = 0,$$

which yields the connection formula

$$y_n(x;\alpha) = \sum_{m=0}^{n} (-1)^m (2m+\beta+1) \times \frac{(-n)_m (n+\alpha+1)_m \Gamma(m+\beta+1)\Gamma(\beta-\alpha+1)}{m!\Gamma(n+m+\beta+2)\Gamma(m-n+\beta-\alpha+1)} y_m(x;\beta).$$

Maple

Thank for your attention