ON ORTHOGONAL POLYNOMIALS WITH RESPECT TO NORMAL OPERATORS

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Preliminaries

(i). Normal operator

(ii). Numerical range

(iii). Spectrum

(iv). Self-adjoint operator

(v). Norm

(v). ζ is a cyclic vector if

$$\overline{span\{T^i\zeta : i = 0, 1, 2, ...\}} = H$$

Functional calculus approach

Consider $f(T)$, where $f : \sigma(T) \to \mathbb{C}$ is a continuous function defined on the spectrum of T. First, let

$$p(z) = \Sigma_{l,k=0}^{n} c_{lk} z^l \overline{z}^k$$

Define $p(T)$ by

$$p(T) = \Sigma_{l,k=0}^{n} c_{lk} T^l T^{*k}$$

Remark 1 *Note that* $TT^* = T^*T$ *is a necessity, since* $p(T)$ *is not well-defined otherwise. Using the Stone- Weierstrass theorem, one can define* $f(T)$ *for arbitrary* $f \in C(\sigma(T))$ *as a limit of operators of the form* $p(T)$ *in the norm topology.*

Proposition 2 *If* $T \in B(H)$ *is a normal operator, then the previously defined map* $C(\sigma(T)) \ni f \mapsto f(T) \in B(H)$ *is a well-defined isometric C*-algebra homeomorphism of* $C(X)$ *onto the C*-algebra generated by* T *and* I*.*

Examples of Normal operators

(i). **Multiplication operators**

Let $H = L^2(X, \mu)$ be a given Hilbert space and let $f \in L^\infty(X, \mu)$.

Then $M_f : L^2(X, \mu) \to L^2(X, \mu)$ defined by $(M_f g)(x) := f(x)g(x)$ is called a multiplication operator.

Known facts on multiplication operators:

1. $\|f\|_\infty = \|M_f\|$
2. $\sigma(M_f) = \{x \in X : \mu(|f - x| < \varepsilon) > 0, \text{ for all } \varepsilon > 0\}$

(ii). **Diffusion operators**

(iii). **Self-adjoint differential operators**

(iv). **Self-adjoint integral operators**

Characterization of normal operators

An operator T is said to be:

(i). **Quasinormal** if $TT^*T = T^*TT$

(ii). **Subnormal** if for a Hilbert space H there is a subspace X of H, and a normal operator $S \in B(H)$ such that $S(X) = X$ and $T = S_{|X}$

(iii). **Hyponormal** if $\|T^*x\| \leq \|Tx\|$, for all $x \in H$

(iv). **Paranormal** if $\|Tx\|^2 = \|T^2x\|$, for all $x \in H$

(v). **Normaloid** if $\|T\| = \sup\{|\lambda| : \lambda \in \sigma(T)\}$
The following implications hold:
$Normal \Rightarrow Quasinormal \Rightarrow Subnormal \Rightarrow$
$\Rightarrow Hyponormal \Rightarrow Paranormal \Rightarrow Normaloid$

Spectral theorem for normal operators

Theorem 3 *For every normal $T \in B(H)$, there exists a finite measure space (X, Ω, μ) and a function $f \in L^\infty(X, \mu)$ such that T is unitarily equivalent to M_f, that is, there is a unitary transformation $U : H \to L^2(X, \mu)$ such that $T = U^{-1} M_f U$.*

Remark 4 *If T does not have a cyclic vector, then we can decompose H into an orthogonal sum of subspaces of H in which there is a cyclic vector for T as per the next lemma*

Lemma 5 *Let* $T \in B(H)$ *be a bounded normal operator. Then there exists a sequence of subspaces* $\{H_i\}_{i \in I}$ *such that for each* i, H *is invariant for* T, *it has a cyclic vector and* $H = \Sigma_{i \in I} H_i$.

Finite dimensional case for spectral theorem

Consider $T : \mathbb{R}^n \to \mathbb{R}^n$ be a self-adjoint operator.

Suppose that its eigenvalues $\lambda_1, ..., \lambda_n$ are distinct

and the corresponding normalized eigenvectors are denoted with $v_1, ..., v_n$,

and they form an orthonormal basis.

Then the spectral measure at the vector v is defined by

$$\mu_v = \Sigma_{k=1}^n |\langle v_k, v\rangle|^2 \delta_{\lambda_k}.$$

Proposition 6 *Let $T \in B(H)$ be normal then for every polynomial p, we have*

$$\langle p(T)v, v\rangle = \int p(x) d\mu_v(x).$$

Proof. We have that $v_1, ..., v_n$ form an orthonormal basis, so $v = \Sigma_{i=1}^n \langle v_i, v\rangle v_i$. Therefore,

$$\begin{aligned} \langle T^k v, v\rangle &= \Sigma_{i,j=1}^n \lambda_i^k |\langle v_i, v\rangle|^2 |\langle v_j, v\rangle|^2 \langle v_i, v_j\rangle \\ &= \Sigma_{i=1}^n \lambda_i^k |\langle v_i, v\rangle|^2 \\ &= \int x^k d\mu_v(x), \end{aligned}$$

for all $k = 1, 2, ...$

$Q.E.D$

Remark 7 *If v is cyclic then μ_v is a spectral measure.*

Orthogonal Polynomials

Classical orthogonal polynomials:
The well known families of orthogonal polynomials are the:

- Jacobi
- Laguerre
- Hermite
- Bessel.

Consider μ to be a finite Borel measure on the real line with compact support K.
We assume that K contains infinitely many points in its support and $\int x^k d\mu_v(x)$ is finite.

Proposition 8 *For every polynomial $\Phi(x)$ for which $\Phi(x) > 0$, $x \in K$, we have $\int \Phi(x) d\mu_v(x)$.*

Theorem 9 *There is a unique sequence of orthonormal polynomials with respect to* μ*,* $\{p_n(x)\}_{n=0}^{\infty}$*, with* $deg(p_n) \leq n$ *such that:*

(i). $p_n(x) = \lambda_n x^n + ...,\ \lambda_n > 0,$

(ii). $\int p_n(x) p_m(x) d\mu(x) = \delta_{m,n}.$

Proof. **Outline of proof**

(i). Consider Gram-Schimdt process.

(ii). Put in the process in (i) $\{1, x, x^2, ...\}$.

(iii). The output will have the required properties.

(iv). Writing every polynomial as a finite linear combination of $p_n - S$ give uniticity.

$Q.E.D$

Remark 10 *The zeroes of* $p_n(x)$ *are simple and real.*

Remark 11 *Orthogonal polynomials on the real line are special in some sense. It is possible to obtain a three-term recurrence formula for them, which plays an essential role in their study.*

Proposition 12 [2] *The polynomials* $p_n(x)$ *satisfy the recurrence formula*

$$xp_n(x) = b_n p_{n+1}(x) + a_n p_n(x) + b_{n-1} p_{n-1}(x),$$

where $p - 1(x) \equiv 0,\ b_n > 0.$

Proposition 13 [5] *Let* $T : \mathbb{R}^n \to \mathbb{R}^n$ *be an invertible self-adjoint linear operator with simple spectrum. Then there is a unique orthonormal basis* $\{v_1, ..., v_n\}$ *of* $\mathbb{R}^n$ *such that the matrix of* T *in this basis is a Jacobi matrix.*

Theorem 14 (Infinite dimensional form) *If* $T : H \to H$ *is a positive normal operator which has a cyclic vector, then there is a basis in which its matrix is a Jacobi matrix.*

Proof: Analogous to proof of [5], Theorem 7.13.

More work on Convolution operators and $\rho-$system

This study continues....
We study the connections between the three systems studied by Musonda [4]

We consider two convolution operators

$$Bf(z) = \int_{-\infty}^{\infty} \frac{f(t)dt}{2cosh\frac{\pi}{2}(z-t)}$$

and

$$Sf(x) = \lim_{G \to 0^+} \int_{|x-t|>G}^{\infty} \frac{f(t)dt}{2sinh\frac{\pi}{2}(x-t)}$$

found in [4].

We need to extend the work of [4] to show that orthogonal polynomials can be used to:

1. Determine L^2-boundedness of operator B and S when they are expressed as eigenvectors
2. Estimate their upper and lower norm bounds when restriction is put to normal operators.

Conclusion

Question: How can we describe eigenvectors and eigenvalues of a **NORMAL** operator?

Hint: When eigenvectors are orthogonal polynomials

References

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THE END

THANK YOU!!!