On certain properties of a perturbed Freud-type weight

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Plan of the talk

Orthogonal polynomials on the real line

- Let $\mathbb{P} = \text{span}\{x^k : k \in \mathbb{N}_0\}$ be the linear space of polynomials with real coefficients and consider the inner product $\langle \cdot, \cdot \rangle_\mu : \mathbb{P} \times \mathbb{P} \to \mathbb{R}$:

$$\langle f, g \rangle_\mu = \int_a^b f(x)g(x)\, d\mu(x), \quad f, g \in \mathbb{P}, \quad \text{supp}(\mu) = [a, b] \subseteq \mathbb{R}.$$

- Let $\{P_n(x)\}_{n=0}^\infty$ be a monic orthogonal polynomial sequence with respect to this inner product:

$$\langle P_m, P_n \rangle_\mu = \int_a^b P_n(x)P_m(x)\, d\mu(x) = \zeta_n \delta_{mn}, \quad \zeta_n > 0.$$

- If μ is absolutely continuous; i.e., $d\mu(x) = w(x)\, dx$, then

$$\mu_k = \int_a^b x^k w(x)\, dx < \infty, \quad k = 0, 1, 2, \ldots,$$

are said to be moments for a positive weight function $w(x)$.

Theorem (Three-term recurrence)

Let $\{P_n(x)\}_{n=0}^{\infty}$ *be a sequence of monic orthogonal polynomials on* $[a,b]$ *relative to an inner product* $\langle \cdot, \cdot \rangle_w$,

$$P_0(x) = 1, \quad P_1(x) = x - \frac{\langle xP_0(x), P_0(x)\rangle_w}{\langle P_0(x), P_0(x)\rangle_w}.$$

Then $\{P_n(x)\}_{n=0}^{\infty}$ *satisfies the recursive scheme*

$$xP_n(x) = P_{n+1}(x) + \alpha_n P_n(x) + \beta_n P_{n-1}(x), \tag{1}$$

where

$$\alpha_n = \frac{\langle xP_n, P_n\rangle}{\|P_n\|^2} = \frac{1}{\zeta_n}\int_{\mathbb{R}} x\, P_n^2(x) w(x)\, dx; \; \beta_n = \frac{1}{\zeta_{n-1}}\int_{\mathbb{R}} x\, P_n(x)\; P_{n-1} w(x)\, dx > 0.$$

- Given the positive measure μ (the weight w), what are the recurrence coefficients?
  For classical orthogonal polynomials (Jacobi, Hermite, Laguerre), their recurrence coefficients are explicit (cf. Chihara, Szegó, Rainville).

Semi-classical orthogonal polynomials

- Classical orthogonal polynomials are characterized by their weight function, which satisfy Pearson's equation

$$[\sigma(x)w(x)]' = \tau(x)w(x), \tag{2}$$

where $\sigma, \tau \in \mathbb{P}$ with $\deg(\sigma) \leq 2$ and $\deg(\tau) = 1$ and boundary conditions $(\sigma w)(a) = 0 = (\sigma w)(b)$, whereas semi-classical orthogonal polynomials have (2) with $\deg(\sigma) > 2$ or $\deg(\tau) \neq 1$ (Hendriksen and van Rossum, 1977).

Weight function	$w(x)$	Parameters	$\sigma(x)$	$\tau(x)$
Semi-classical Laguerre	$x^{\lambda} exp(-x^2 + tx)$	$\lambda > -1$	x	$1 + \lambda + tx - 2x^2$
Freud	$exp(-\frac{1}{4}x^4 - tx^2)$	$x, t \in \mathbb{R}$	1	$-2tx - x^3$
Generalized Freud	$\lvert x\rvert^{2\lambda+1} exp(-x^4 + tx^2)$	$\lambda > 0,\ x, t \in \mathbb{R}$	x	$2\lambda + 2 - 2tx^2 - x^4$

- These polynomials satisfy a structural relation (Maroni, 1985)

$$\sigma(x)P'_{n+1}(x) = \sum_{j=n-s}^{n+r} A_{n,j}P_j(x), \quad \begin{cases} r = \deg(\sigma), \\ s = \max\{\deg(\sigma) - 2, \deg(\tau) - 1\} \end{cases} .$$

- In this case, the recurrence coefficients are usually not explicit and they obey non-linear recurrence relations.

Certain semi-classical weights obey non-linear recurrence Equations

- **The weight $w(x) = \exp(-x^4)$ on $\mathbb{R}$ (cf. Nevai, 1983):**
  Since $\alpha_n = 0$ (symmetry) in the ttrr (1), the coefficient β_n obeys

$$4\beta_n\left(\beta_{n-1} + \beta_n + \beta_{n+1}\right) = n; \quad \beta_0 = 0, \quad \beta_1 = \frac{\int_{-\infty}^{\infty} x^2 \exp(-x^4)\,dx}{\int_{-\infty}^{\infty} \exp(-x^4)\,dx} = \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}.$$

- **The semiclassical Laguerre $w_\nu(x) = x^\nu \exp(-x^2 + tx),\ \nu > -1,\ x \in \mathbb{R}^+$:**
  (W. Van Assche, L. Boelen (2011) and (P. Clarkson, K. H. Jordaan (2014))

$$\left.\begin{aligned} (2\alpha_n - t)(2\alpha_{n-1} - t) &= \frac{(2\beta_n - n)(2\beta_n - n - \nu)}{\beta_n}, \\ 2\beta_n + 2\beta_{n+1} - \alpha_n(2\alpha_n - t) &= 2n + 1 + \nu. \end{aligned}\right\}$$

- **The Freud weight $w(x) = \exp\left(-x^4 + tx^2\right),\ x \in \mathbb{R}$ (Freud (1976)):**

$$\frac{n}{\beta_n} = -2t + 4\left[\beta_{n+1} + \beta_n + \beta_{n-1}\right], \quad \beta_0 = 0, \tag{3}$$

  (3) is known as Shohat-Freud's ('String' or Discrete Painlevé) equation.

- **Asymptotic behavior**: $\beta_n = \left(\frac{n}{12}\right)^{1/2}\left[1 + \frac{1}{24n^2} + \mathcal{O}(n^{-4})\right]$ (Nevai, 1984).

The link to Painlevé equations

- **Some history**: The first non-linear recurrence equation - **Shohat** (1930's) and **Laguerre**, **Freud** (late 70's) and very recently recognized as discrete Painlevé equations by **Fokas, Its,** and **Kitaev**. Work by **Magnus** (relation between discrete and continuous Painlevé equations), **Witte**, **Clarkson**, **Van Assche**, **Nijhoff**, **Spicer**, **Chen** and **Ismail** extended theory with some more examples.
- **The Painlevé equations** are a chapter in '**DLMF**'.

$$\frac{d^2q}{dt^2} = 6q^2 + z,\ z \in \mathbb{C}\ \ (\mathrm{P_I});\quad \mathrm{P_{II}}(\alpha) : \frac{d^2q}{dz^2} = 2q^3 + zq + \alpha,$$
$$\mathrm{P_{III}}(\alpha,\beta,\gamma,\delta) : \frac{d^2q}{dz^2} = \frac{1}{q}\left(\frac{dq}{dz}\right)^2 - \frac{1}{z}\frac{dq}{dz} + \frac{1}{z}(\alpha q^2 + \beta) + \gamma q^3 + \frac{\delta}{q},$$
$$\mathrm{P_{IV}}(\alpha,\beta) : \frac{d^2q}{dz^2} = \frac{1}{2q}\left(\frac{dq}{dz}\right)^2 + \frac{3}{2}q^3 + 4zq^2 + 2(z^2-\alpha)q + \frac{\beta}{q}, \tag{4}$$

where $\alpha, \beta, \gamma, \delta$ are constants and $z \in \mathbb{C}$.

- **Some discrete Painlevé equations:**

$$(\text{d-P}_{\mathrm{I}})\quad x_{n+1} + x_n + x_{n-1} = \frac{z_n + \gamma(-1)^n}{x_n} + \sigma$$
$$(\text{d-P}_{\mathrm{II}})\quad x_{n+1} + x_{n-1} = \frac{x_n z_n + \gamma}{1 - x_n^2}$$
$$(\text{d-P}_{\mathrm{IV}})\quad (x_{n+1} + x_n)(x_n + x_{n-1}) = \frac{\left(x_n^2 - \kappa^2\right)\left(x_n^2 - \mu^2\right)}{(x_n + z_n)^2 - \gamma^2}$$

Certain semi-classical weights giving rise to (discrete) Painlevé equations

- $w(x) = |x|^{\varrho} \exp(-x^4)$, $\varrho > -1$ on $\mathbb{R}$ is related to $\mathrm{dP_I}$ (Magnus, 1986).
- $w(x;t) = \exp(-\frac{1}{4}x^4 + tx^2)$, $t \in \mathbb{R}$ on $\mathbb{R}$ is related to $\mathrm{dP_I}$ (Magnus, 1995).

In the continuous sense,

- $w(x) = x^{\lambda} \exp(-x^2 + tx)$ on $\mathbb{R}^+$ is related to $\mathrm{P_{IV}}$ (Galina et. al, Clarkson et.al).
- $w(x,t) = (1-x)^{\alpha}(1+x)^{\beta} \exp(-tx)$, $\alpha, \beta > -1$, $x \in [-1,1]$, $t \in \mathbb{R}$ related to $\mathrm{P_V}$, (Basor, Chen and Ehrhardt (2009)).
- $w(x,t) = x^{\alpha} \exp(-x - t/x)$, $\alpha > -1$, $x \in \mathbb{R}^+$, related to $\mathrm{P_{III}}$, Chen & Its (2010).

Some observations

- Solutions of Painlevé are sometimes not directly the recurrence coefficients, but functions of these, with extra terms and/or changes of variable.
  For e.g. (after Clarkson & Jordaan), if $w(x;t) = x^{\lambda} \exp\left(-x^2 + tx\right)$, $\lambda > -1$ then the function $q_n(z) = 2\alpha_n(t) + t$, with $z = \frac{1}{2}t$ satisfies $\mathrm{P_{IV}}$ in z & $(A,B) = (2n + \lambda + 1, -2\lambda^2)$.
- The solutions obtained from deforming OPs are typically not 'generic', but with **very specific values of the parameters**.
- Special function solutions of $\mathrm{P_{IV}}$ are expressed in terms of **parabolic cylinder** functions.

Ladder relations help to obtain nonlinear equations

- Assume the weight w vanishes at the endpoints of $[a,b] \subseteq \mathbb{R}$. The ladder operators for the polynomials $P_n(z)$ (cf. Chen & Ismail, Van Assche et.al) are given by

$$\left.\begin{aligned} \left[\tfrac{d}{dz} + B_n(z)\right] P_n(z) = \beta_n A_n(z) P_{n-1}(z) \\ \left[\tfrac{d}{dz} - B_n(z) - \nu'(z)\right] P_{n-1}(z) = -A_{n-1}(z) P_n(z) \end{aligned}\right\} \tag{5}$$

  where $\nu(x) = -\log w(x)$, since $w(x) > 0,\ x \in [a,b] \subseteq \mathbb{R}$ and

- the coefficients in (5) are given by

$$A_n(z) = \frac{1}{h_n} \int_{-\infty}^{\infty} P_n^2(y) \left[\frac{\nu'(z) - \nu'(y)}{z - y}\right] w(y)\, dy,$$

$$B_n(z) = \frac{1}{h_{n-1}} \int_{-\infty}^{\infty} P_n(y) P_{n-1}(y) \left[\frac{\nu'(z) - \nu'(y)}{z - y}\right] w(y)\, dy,$$

Note: We can calculate $A_n(z)$ and $B_n(z)$ without explicitly knowing the polynomials other than the weight function.

Compatibility conditions (cf. Chen & Ismail, Magnus, Van Assche et.al),

For the ladder operators, the associated compatibility conditions are

Lemma

- *The functions $A_n(z)$ and $B_n(z)$ satisfy*

$$B_{n+1}(z) + B_n(z) = (z - \alpha_n)\, A_n(z) - \nu'(z)$$
$$1 + (z - \alpha_n)\,(B_{n+1}(z) - B_n(z)) = \beta_{n+1} A_{n+1}(z) - \beta_n A_{n-1}(z),$$

valid for all $z \in \mathbb{C} \cup \{\infty\}$.

- *The functions $A_n(z)$, $B_n(z)$ and $\sum_{k=0}^{n-1} A_k(z)$ satisfy the identity*

$$B_n^2(z) + \nu'(z) B_n(z) + \sum_{k=0}^{n-1} A_k(z) = \beta_n A_n(z) A_{n-1}(z).$$

Extract from Digital Library of Mathematical Functions

§18.32 Orthogonal polynomials with Respect to Freud Weights

A *Freud weight* is a weight function of the form

18.32.1 $$w(x) = exp(-Q(x)), \quad -\infty < x < \infty$$

where $Q(x)$ is real, even, non-negative, and continuously differentiable. Of special interest are the cases $Q(x) = x^{2m}$, $m = 1, 2, \ldots$. *No explicit expressions for the corresponding OP's are available*. However, for asymptotic approximations in terms of elementary functions for the OP's, and also for their largest zeros, see Levin and Lubinsky [2001]. For a uniform asymptotic expansion in terms of Airy functions for the OP's in the case x^4 see Bo and Wong [1999].

Our interest:

Can we obtain concise formulations for the recurrence coefficients as well as the polynomials that are orthogonal with respect to the generalized Freud weight? What more non-linear recurrence relations can one obtain?

A 'Generalized Freud' Weight

- Monic orthogonal polynomials with respect to the generalized Freud inner product

$$\langle p,q\rangle = \int_{\mathbb{R}} p(x)q(x)|x|^{2\lambda+1}\exp(-x^4+tx^2)\,dx, \quad x\in\mathbb{R},\ (t\in\mathbb{R}),\ \lambda>0, \tag{7}$$

  satisfy the three-term recurrence relation

$$xS_n(x;t) = S_{n+1}(x;t) + \beta_n(t;\lambda)S_{n-1}(x;t), \quad n=1,2,\ldots, \tag{8}$$

  where initial conditions $S_{-1}\equiv 0$, $S_0\equiv 1$ and $\beta_n(t;\lambda)>0$.

- The coefficients $\beta_n(t;\lambda)$ satisfy a Toda-type equation as in below.

Theorem (Aptekarev et. al)

Let μ be a symmetric positive measure on $\mathbb{R}$ for which all the moments exist and let μ_t be the measure for which $d\mu_t(x) = \exp(tx^2)\,d\mu(x)$, where $t\in\mathbb{R}$ is such that all the moments of μ_t exist. Then the recurrence coefficients of the orthogonal polynomials for μ_t satisfy the differential-difference equation

$$\frac{d}{dt}\beta_n = \beta_n\left[\beta_{n+1}-\beta_{n-1}\right], \quad n=1,2,\ldots. \tag{9}$$

Moments for the generalized Freud weight

Lemma (The first moment)

The first moment, $\mu_0(t;\lambda)$, for generalized Freud weight is given by

$$\mu_0(t;\lambda) = \frac{\Gamma(\lambda+1)}{2^{(\lambda+1)/2}} \exp\left(\tfrac{1}{8}t^2\right) D_{-\lambda-1}\left(-\tfrac{1}{2}\sqrt{2}\,t\right),$$

where

$$D_\nu(\xi) = \frac{\exp(-\frac{1}{4}\xi^2)}{\Gamma(-\nu)} \int_0^\infty s^{-\nu-1} \exp\left(-\tfrac{1}{2}s^2 - \xi s\right) ds, \quad \mathrm{Re}(\nu) < 0,$$

is a parabolic cylinder (Hermite-Weber) function.

This result follows from the definition of the first moment

$$\mu_0(t;\lambda) = \int_{-\infty}^{\infty} |x|^{2\lambda+1} \exp\left(-x^4 + tx^2\right) dx$$

and the integral representation of the parabolic cylinder function.

Higher-order moments

The even moments are

$$\begin{aligned}\mu_{2n}(t;\lambda) &= \int_{-\infty}^{\infty} x^{2n}\,|x|^{2\lambda+1} \exp\left(-x^4+tx^2\right) dx \equiv \mu_0(t;\lambda+n)\\ &= \frac{d^n}{dt^n}\left(\int_{-\infty}^{\infty} |x|^{2\lambda+1} \exp\left(-x^4+tx^2\right)\, dx\right)\\ &= \frac{d^n}{dt^n}\mu_0(t;\lambda), \quad n=1,2,\ldots,\end{aligned}$$

whilst the odd ones are

$$\mu_{2n+1}(t;\lambda) = \int_{-\infty}^{\infty} \underbrace{x^{2n+1}\,|x|^{2\lambda+1} \exp\left(-x^4+tx^2\right)}_{Odd} dx = 0, \quad n=1,2,\ldots,$$

since the integrand is odd. When $\lambda = n \in \mathbb{Z}^+$,

$$D_{-n-1}\ \left(-\frac{1}{2}\sqrt{2}t\right) = \frac{1}{2}\sqrt{2\pi}\ \ \frac{d^n}{dt^n}\left(\left[1+\mathrm{erf}(\frac{1}{2}t)\right]\exp(\frac{1}{8}t^2)\right),$$

with $\mathrm{erfc}(z)$ is the error function.

Recurrence Coefficients (Concise form)

Theorem (Concise formulation of $\beta_n(t;\lambda)$ in terms of Tau function)

The recurrence coefficients $\{\beta_n(t;\lambda)\}_{n=1}^{\infty}$ in the three-term recurrence relation

$$xS_n(x;t) = S_{n+1}(x;t) + \beta_n(t;\lambda)S_{n-1}(x;t), \;\; n \in \mathbb{N},$$

are explicitly given by

$$\beta_{2n}(t;\lambda) = \frac{d}{dt}\ln\frac{\tau_n(t;\lambda+1)}{\tau_n(t;\lambda)}; \quad \beta_{2n+1}(t;\lambda) = \frac{d}{dt}\ln\frac{\tau_{n+1}(t;\lambda)}{\tau_n(t;\lambda+1)}, \tag{10}$$

where $\tau_n(t;\lambda)$ is the Wronskian given by

$$\tau_n(t;\lambda) = \mathcal{W}\left(\phi_\lambda, \frac{d\phi_\lambda}{dt}, \ldots, \frac{d^{n-1}\phi_\lambda}{dt^{n-1}}\right) = \det\left[\frac{d^{j+k}}{dt^{j+k}}\mu_0(t;\lambda)\right]_{j,k=0}^{n-1},$$

with $\tau_0(t;\lambda) = 1$ and $\phi_\lambda(t) = \mu_0(t;\lambda) = \frac{\Gamma(\lambda+1)}{2^{(\lambda+1)/2}}\exp\left(\frac{1}{8}t^2\right)D_{-\lambda-1}\left(-\frac{1}{2}\sqrt{2}\,t\right)$.

Remark: The function $H_n(t;\lambda) := \frac{d}{dt}\ln\tau_n(t;\lambda)$ satisfies the 2nd order, 2nd degree equation

$$4\left(\frac{d^2H_n}{dt^2}\right)^2 - \left(t\frac{dH_n}{dt} - H_n\right)^2 + 4\frac{dH_n}{dt}\left(2\frac{dH_n}{dt} - n\right)\left(2\frac{dH_n}{dt} - n - \lambda\right) = 0.$$

which is equivalent to $\mathrm{S_{IV}}$, the $\mathrm{P_{IV}}$ σ-equation and so the coefficients take the form

$$\beta_{2n}(t;\lambda) = H_n(t;\lambda+1) - H_n(t;\lambda); \quad \beta_{2n+1}(t;\lambda) = H_{n+1}(t;\lambda) - H_n(t;\lambda+1).$$

Sample recurrence coefficients in terms of Φ_λ

The first few recurrence coefficients $\beta_n(t;\lambda)$ are given by

$$\begin{aligned}\beta_1(t;\lambda) &= \Phi_\lambda,\\ \beta_2(t;\lambda) &= -\frac{2\Phi_\lambda^2 - t\Phi_\lambda - \lambda - 1}{2\Phi_\lambda},\\ \beta_3(t;\lambda) &= -\frac{\Phi_\lambda}{2\Phi_\lambda^2 - t\Phi_\lambda - \lambda - 1} - \frac{\lambda+1}{2\Phi_\lambda},\end{aligned}$$

where

$$\begin{aligned}\Phi_\lambda(t) &= \frac{d}{dt}\ln\left\{D_{-\lambda-1}\left(-\tfrac{1}{2}\sqrt{2}\,t\right)\exp\left(\tfrac{1}{8}t^2\right)\right\}\\ &= \tfrac{1}{2}t + \tfrac{1}{2}\sqrt{2}\,\frac{D_{-\lambda}\left(-\tfrac{1}{2}\sqrt{2}\,t\right)}{D_{-\lambda-1}\left(-\tfrac{1}{2}\sqrt{2}\,t\right)}.\end{aligned}$$

The first few polynomials

By using the recurrence relation

$$xS_n(x;t) = S_{n+1}(x;t) + \beta_n(t;\lambda)S_{n-1}(x;t),$$

with

$$S_0(x;t) = 1,\ S_0(x;t) = 0,\ \beta_0(t;\lambda) = 0,$$

then the first few polynomials are given by

$$\begin{aligned}
S_1(x;t) &= x,\\
S_2(x;t) &= x^2 - \Phi_\lambda,\\
S_3(x;t) &= x^3 - \frac{t\Phi_\lambda + \lambda + 1}{2\Phi_\lambda}\,x,\\
S_4(x;t) &= x^4 - \frac{2t\Phi_\lambda^2 - (t^2+2)\Phi_\lambda - (\lambda+1)t}{2(2\Phi_\lambda^2 - t\Phi_\lambda - \lambda - 1)}\,x^2\\
&\qquad - \frac{2(\lambda+2)\Phi_\lambda^2 - (\lambda+1)t\Phi_\lambda - (\lambda+1)^2}{2(2\Phi_\lambda^2 - t\Phi_\lambda - \lambda - 1)}.
\end{aligned}$$

Linking $\beta_n(t;\lambda)$ to the general $\mathrm{dP_I}$

Theorem (Link of the coefficient $\beta_n(t;\lambda)$ to $\mathrm{dP_I}$)

The recurrence coefficients associated with the generalized Freud weight $w_\lambda(x;t)$ satisfy the nonlinear difference equation; i.e., the general discrete $\mathrm{dP_I}$:

$$\boxed{\beta_{n+1}+\beta_n+\beta_{n-1}=\tfrac{1}{2}t+\frac{2n+(2\lambda+1)[1-(-1)^n]}{8\beta_n},}$$

where $\beta_0=0$ and β_1 is given by

$$\beta_1(t;\lambda)=\frac{\mu_2(t;\lambda)}{\mu_0(t;\lambda)}=\frac{\int_{-\infty}^{\infty}x^2|x|^{2\lambda+1}\exp\left(-x^4+tx^2\right)dx}{\int_{-\infty}^{\infty}|x|^{2\lambda+1}\exp\left(-x^4+tx^2\right)dx}$$
$$=\tfrac{1}{2}t+\tfrac{1}{2}\sqrt{2}\,\frac{D_{-\lambda}\left(-\tfrac{1}{2}\sqrt{2}\,t\right)}{D_{-\lambda-1}\left(-\tfrac{1}{2}\sqrt{2}\,t\right)}.$$

Sketch of the proof:

- Recall that $\zeta_n = \beta_1 \cdots \beta_n$. Consider the integral

$$J_n := \int_{\mathbb{R}} [P_{n-1}(y)P_n(y)]' w_\lambda(y)dy.$$

- By using orthogonality, we obtain $J_n = n\zeta_{n-1}$.
- Integrating by parts yields

$$\begin{aligned} J_n = n\zeta_{n-1} &= \int_{\mathbb{R}} \left(4y^3 - 2ty - \frac{2\lambda+1}{y}\right) P_n(y)P_{n-1}(y)w_\lambda(y)dy \\ &= \int_{\mathbb{R}} [4y^2 - 2t]\left[P_{n+1}(y) + \beta_n P_{n-1}(y)\right] P_{n-1}(y)w_\lambda(y)dy \\ &\quad - \int_{\mathbb{R}} \frac{2\lambda+1}{y} P_n(y)P_{n-1}(y)w_\lambda(y)dy \\ &= -2t\beta_n\zeta_{n-1} + 4\zeta_{n+1} + 4\beta_n \int_{\mathbb{R}} (P_n(y) + \beta_{n-1}P_{n-1}(y))^2 w_\lambda(y)dy - \gamma\Omega_n\zeta_{n-1} \\ &= -2t\beta_n\zeta_{n-1} + 4\zeta_{n+1} + 4\beta_n[\zeta_n + \beta_{n-1}^2\zeta_{n-2}] - \gamma\Omega_n\zeta_{n-1}, \quad \gamma = 2\lambda+1. \end{aligned}$$

This yields $\mathrm{dP_I}$.

Linking $\beta_n(t;\lambda)$ to continuous $\mathrm{P_{IV}}$

Theorem

The coefficient $\beta_n(t;\lambda)$ in equation (8) satisfy the nonlinear differential equation

$$\frac{d^2\beta_n}{dt^2} = \frac{1}{2\beta_n}\left(\frac{d\beta_n}{dt}\right)^2 + \tfrac{3}{2}\beta_n^3 - t\beta_n^2 + (\tfrac{1}{8}t^2 - \tfrac{1}{2}A_n)\beta_n + \frac{B_n}{16\beta_n}, \tag{11}$$

which is equivalent to $\mathrm{P_{IV}}$, where the parameters A_n and B_n are given by

$$\begin{pmatrix} A_{2n} \\ B_{2n} \end{pmatrix} = \begin{pmatrix} -2\lambda - n - 1 \\ -2n^2 \end{pmatrix}; \quad \begin{pmatrix} A_{2n+1} \\ B_{2n+1} \end{pmatrix} = \begin{pmatrix} \lambda - n \\ -2(\lambda + n + 1)^2 \end{pmatrix}.$$

Remark: Equation (11) $\equiv \mathrm{P_{IV}}$ through $\beta_n(t;\lambda) = \frac{1}{2}\nu(z)$, with $z = -\frac{1}{2}t$. Hence

$$\beta_{2n}(t;\lambda) = \tfrac{1}{2}\nu\left(z; -2\lambda - n - 1, -2n^2\right), \quad \beta_{2n+1}(t;\lambda) = \tfrac{1}{2}\nu\left(z; \lambda - n, -2(\lambda + n + 1)^2\right), \tag{12}$$

with $z = -\frac{1}{2}t$, where $\nu(z; A, B)$ satisfies $\mathrm{P_{IV}}$.

Sketch of the proof:

- Consider the $\mathrm{dP_I}$ with $\beta_n = x_n$; i.e.,

$$n + (2\lambda + 1)\Omega_n = 4x_n\left(x_{n+1} + x_n + x_{n-1} - \frac{1}{2}t\right), \tag{13}$$

- Combine (13) with the Toda equation (9); that is, $x_n' = x_n(x_{n+1} - x_{n-1})$,
- and finally eliminate x_{n+1} and x_{n-1} to obtain $\mathrm{P_{IV}}$ as in (11).

Ladder relations for the case generalized Freud

Theorem

The monic orthogonal polynomials $S_n(x;t)$ with respect to the generalized Freud weight on $\mathbb{R}$ satisfy the differential-difference recurrence relation

$$S_n'(x;t) = \beta_n A_n(x;t) S_{n-1}(x;t) - B_n(x;t) S_n(x;t) \tag{14}$$

where

$$A_n(x;t) = 4x^2 + 4(\beta_n + \beta_{n+1}) - 2t,$$
$$B_n(x;t) = 4x\beta_n + \left(\frac{2\lambda+1}{x}\right)\Omega_n,$$

where the expression Ω_n is given by $\Omega_n = \dfrac{1-(-1)^n}{2}$.

Equation (14) is also known as **Structural relation** or **Lowering operators**.

(More) non-linear Difference equations

Theorem (Non-linear equations)

For the weight in (7), the coefficients β_n in (8) satisfy the following system of equations:

$$\beta_{n+1}\left[4\beta_{n+2}+4\beta_{n+1}-2t\right]-\beta_n\left[4\beta_{n+1}+4\beta_n-2t\right]=(2\lambda+1)\left[\Omega_{n+1}-\Omega_n\right]+1,$$

$$\sum_{k=0}^{n-1}(\beta_k+\beta_{k+1})=16\beta_n^2\Big[(\beta_{n-1}+\beta_n+\beta_{n+1}-t)+\left[(8\beta_{n+1}-4t)(2\beta_{n-1}-t)\right]$$
$$+2t\left[n+(2\lambda+1)\Omega_n+n\right]\Big],$$

Theorem (Differential-recurrence)

For the generalized Freud weight, the recurrence coefficients $\beta_n=\beta_n(t)$ satisfy

$$\frac{d\beta_n}{dt}=\beta_n\left(\beta_{n+1}-\beta_{n-1}\right),$$
$$\frac{d^2\beta_n}{dt^2}=\beta_n\Big[\beta_{n+1}\left(\beta_{n+1}+\beta_{n+2}-2\beta_{n-1}-\beta_n\right)+\beta_{n-1}\left(\beta_{n-2}+\beta_{n-1}-\beta_n\right)\Big].$$

The differential equation

Theorem

For the generalized Freud weight, the monic orthogonal polynomials $S_n(x;t)$ satisfy the differential equation

$$x\frac{d^2S_n}{dx^2}(x;t)+R_n(x;t)\frac{dS_n}{dx}(x;t)+T_n(x;t)S_n(x;t)=0,$$

where the coefficients are given by

$$R_n(x;t)=x\left(-4x^3+2tx+\frac{2\lambda+1}{x}-\frac{2x}{x^2-\frac{1}{2}t+\beta_n+\beta_{n+1}}\right),$$

$$\begin{aligned}T_n(x;t)=x\Big[&4nx^2+4\beta_n+16\beta_n(\beta_n+\beta_{n+1}-\tfrac{t}{2})(\beta_n+\beta_{n-1}-\tfrac{t}{2})\\&-\frac{8\beta_nx^2+(2\lambda+1)[1-(-1)^n]}{x^2-\frac{1}{2}t+\beta_n+\beta_{n+1}}+4(2\lambda+1)(-1)^n\beta_n\\&+(2\lambda+1)[1-(-1)^n]\left(t-\frac{1}{2x^2}\right)\Big].\end{aligned}$$

- Note that one can find a similar result for Freud-like weights (cf. A. Arceo, E.J. Huertas & F. Marcellán (2016)).

Conclusions

- The recurrence coefficients associated with the generalized Freud weight can be expressed in terms of Wronskians of parabolic cylinder functions that arise in the description of special function solutions of the $\mathrm{P_{IV}}$ equation.
- The moments of the generalized Freud weight provide the link between the orthogonal polynomials and the associated Painlevé equation. A concise formulation of the generalized Freud polynomials has also been obtained.
- Part of the results of this work illustrate the increasing significance of the Painlevé equations in the field of orthogonal polynomials and special functions.

Future perspectives

- Investigation of a class of polynomials orthogonal with respect to a more general Shohat-Freud type weight function.
- Certain properties for polynomials orthogonal with respect to a perturbed Airy-type and Hexic-Freud weights are under investigation:
  (NB: The weight $\exp\left(-x^3 + tx\right)$ ('deformed Airy type') as a modification of $exp(-x^3)$ with an exponential factor $\exp(tx)$ is considered by Van Assche, Filipuk and Zhang (2015)).

Thank you!