Summation formula for generalized discrete q-Hermite II polynomials

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By
Sama Arjika
Faculty of Sciences and Technics
University of Agadez, Niger

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Introduction and motivation

The classical orthogonal polynomial (COP) and the quantum orthogonal polynomials (QOP) (also called q-orthogonal polynomials) constitute an interesting set of special functions. They appear in

1. several branches of sciences such as : continued fractions, Eulerian series, theta functions, elliptic functions,$\cdots$ [Andrews (1986), Fine (1988)],
2. quantum groups and quantum algebras [Gasper and Rahman (1990), Koornwinder (1990) and (1994), Nikiforov *et al* (1991), Vilenkin and Klimyk (1992)].

They have been intensively studied in the last years by several people, [Koekoek and Swarttouw (1998), Lesky (2005), Koekoek *et al* (2010)], $\cdots$.

Introduction and motivation

Each family of COP and QOP occupy different levels within the so-called, Askey-Wilson scheme and are characterized by the properties :

1. they are solutions of a hypergeometric second order differential equation,
2. they are generated by a recursion relation,
3. they are orthogonal with respect to a weight function,
4. they obey the Rodrigues-type formula.

In this scheme, the Hermite polynomials are the ground level and most of there properties can be generalized.

Introduction and motivation

In their paper, Àlvarez-Nodarse et al [Int. J. Pure. Appl. Math. **10** (3) 331-342 (2014)], have introduced a q-extension of the discrete q-Hermite II polynomials as :

$$\begin{aligned} \mathcal{H}_{2n}^{(\mu)}(x;q): &= (-1)^n (q;q)_n \, L_n^{(\mu-1/2)}(x^2;q) \\ \mathcal{H}_{2n+1}^{(\mu)}(x;q): &= (-1)^n (q;q)_n \, x \, L_n^{(\mu+1/2)}(x^2;q) \end{aligned} \tag{1}$$

where $\mu > -1/2$, $L_n^{(\alpha)}(x;q)$ are the q-Laguerre polynomials given by

$$L_n^{(\alpha)}(x;q) := \frac{(q^{\alpha+1};q)_n}{(q;q)_n} \, {}_1\phi_1 \left(\begin{array}{c} q^{-n} \\ q^{\alpha+1} \end{array} \middle| q; -q^{n+\alpha+1} x \right). \tag{2}$$

For $\mu = 0$ in (1), the polynomials $\mathcal{H}_n^{(0)}(x;q)$ correspond to the discrete q-Hermite II polynomial

$$\mathcal{H}_n^{(0)}(x;q^2) = q^{n(n-1)} \tilde{h}_n(x;q).$$

Introduction and motivation

Àlvarez-Nodarse et al showed that the polynomials $\mathcal{H}_n^{(\mu)}(x;q)$ satisfy the orthogonality relation

$$\int_{-\infty}^{\infty} \mathcal{H}_n^{(\mu)}(x;q)\mathcal{H}_m^{(\mu)}(x;q)\omega(x)dx = \pi\, q^{-n/2}(q^{1/2};q^{1/2})_n(q^{1/2};q)_{1/2}\,\delta_{nm}$$

on the whole real line $\mathbb{R}$ with respect to the positive weight function $\omega(x) = 1/(-x^2;q)_\infty$. A detailed discussion of the properties of the polynomials $\mathcal{H}_n^{(\mu)}(x;q)$ can be found in [Int. J. Pure. Appl. Math. **10** (3) 331-342 (2014)].

Introduction and motivation

Recently, Saley Jazmat et al [Bulletin of Mathematical Ana. App. **6**(4), 16-43 (2014)], introduced a novel extension of discrete q-Hermite II polynomials by using new q-operators. This extension is defined as :

$$\tilde{h}_{2n,\alpha}(x;q) = (-1)^n q^{-n(2n-1)} \frac{(q;q)_{2n}}{(q^{2\alpha+2};q^2)_n} L_n^{(\alpha)}\left(x^2 q^{-2\alpha-1};q^2\right)$$

$$\tilde{h}_{2n+1,\alpha}(x;q) = (-1)^n q^{-n(2n+1)} \frac{(q;q)_{2n+1}}{(q^{2\alpha+2};q^2)_{n+1}} x\, L_n^{(\alpha+1)}\left(x^2 q^{-2\alpha-1};q^2\right). \tag{3}$$

For $\alpha = -1/2$ in (3), the polynomials $\tilde{h}_{n,-\frac{1}{2}}(x;q)$ correspond to the discrete q-Hermite II polynomials, i.e.,

$$\tilde{h}_{n,-\frac{1}{2}}(x;q) = \tilde{h}_n(x;q).$$

Introduction and motivation

The generalized discrete q-Hermite II polynomials $\tilde{h}_{n,\alpha}(x;q)$ satisfy the orthogonality relation

$$\int_{-\infty}^{+\infty} \tilde{h}_{n,\alpha}(x;q)\tilde{h}_{m,\alpha}(x;q)\omega_\alpha(x;q)|x|^{2\alpha+1}d_qx$$

$$= \frac{2q^{-n^2}(1-q)(-q,-q,q^2;q^2)_\infty}{(-q^{-2\alpha-1},-q^{2\alpha+3},q^{2\alpha+2};q^2)_\infty}\frac{(q;q)_n^2}{(q;q)_{n,\alpha}}\delta_{n,m} \qquad (4)$$

on the real line $\mathbb{R}$ with respect to the positive weight function $\omega_\alpha(x) = 1/(-q^{-2\alpha-1}x^2;q^2)_\infty$.

Introduction and motivation

Motivated by Saley Jazmat's work [Bul. Math. Anal. App. **6**(4), 16-43 (2014)], our interest in this work is

1. to introduce new family of "*generalized discrete q-Hermite II polynomials (in short gdq-H2P)* $\tilde{h}_{n,\alpha}(x, y|q)$" which is an extension of the generalized discrete q-Hermite II polynomials $\tilde{h}_{n,\alpha}(x; q)$,
2. and investigate summation formula.

Outline

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1. Notations and definitions
2. Generalized discrete q-Hermite II polynomials $\{\tilde{h}_{n,\alpha}(x,y|q)\}_{n=0}^{\infty}$
3. Connection formulae for the generalized discrete q-Hermite II polynomials $\{\tilde{h}_{n,\alpha}(x,y|q)\}_{n=0}^{\infty}$

Notations and definitions

Throughout this paper, we assume that $0 < q < 1$, $\alpha > -1$. For a complex number a,

- ★ the q-shifted factorials are defined by :

$$(a;q)_0 = 1;\ (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k),\ n \geq 1;\ (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

- ★ The q-number is defined by :

$$[n]_q = \frac{1-q^n}{1-q}, \qquad n!_q := \prod_{k=1}^{n}[k]_q, \quad 0!_q := 1,\ n \in \mathbb{N}. \tag{5}$$

Notations and definitions

Hahn q-addition and q-subtraction

For $x,\ y \in \mathbb{R}$,

★ the Hahn q-addition $\oplus_q$ is defined by :

$$\begin{aligned}(x \oplus_q y)^n : &= (x+y)(x+qy)\dots(x+q^{n-1}y) \\ &= (q;q)_n \sum_{k=0}^{n} \frac{q^{\binom{k}{2}} x^{n-k} y^k}{(q;q)_k (q;q)_{n-k}}, \quad n \geq 1, \end{aligned} \tag{6}$$

and $(x \oplus_q y)^0 := 1$.

★ The q-subtraction $\ominus_q$ is given by

$$(x \ominus_q y)^n := (x \oplus_q (-y))^n \tag{7}$$

and $(x \ominus_q y)^0 := 1$.

Notations and definitions

1. The generalized backward and forward q-derivative operators $D_{q,\alpha}$ and $D^{+}_{q,\alpha}$, Saley Jazmat et al are defined :

$$D_{q,\alpha}f(x)=\frac{f(x)-q^{2\alpha+1}f(qx)}{(1-q)x},\; D^{+}_{q,\alpha}f(x)=\frac{f(q^{-1}x)-q^{2\alpha+1}f(x)}{(1-q)x}.$$

2. Remark that, for $\alpha=-\frac{1}{2}$, we have $D_{q,\alpha}=D_q$, $D^{+}_{q,\alpha}=D^{+}_q$ where D_q and D^{+}_q are the Jackson's q-derivative with

$$D_qf(x)=\frac{f(x)-f(qx)}{(1-q)x},\; D^{+}_qf(x)=\frac{f(q^{-1}x)-f(x)}{(1-q)x}. \tag{8}$$

3. For $f(x)=x^n$, we have

$$D_{q,\alpha}x^n=[n]_{q,\alpha}x^{n-1},\quad D^{+}_{q,\alpha}x^n=q^{-n}[n]_{q,\alpha}x^{n-1}$$

where $[n]_{q,\alpha}:=[n+2\alpha+1]_q,\quad [n]_{q,-1/2}=[n]_q$.

Generalized q-shifted factorials

The generalized q-shifted factorials are defined as :

$$(n+1)!_{q,\alpha} = [n+1+\theta_n(2\alpha+1)]_q \, n!_{q,\alpha} \tag{9}$$

$$(q;q)_{n+1,\alpha} = (1-q)[n+1+\theta_n(2\alpha+1)]_q (q;q)_{n,\alpha}, \tag{10}$$

where

$$\theta_n = \begin{cases} 1 & \text{if n even} \\ 0 & \text{if n odd.} \end{cases}$$

★ Remark that, for $\alpha = -1/2$, we have

$$(q;q)_{n,-1/2} = (q;q)_n, \quad n!_{q,-1/2} = \frac{(q;q)_n}{(1-q)^n}. \tag{11}$$

★ We denote

$$(q;q)_{2n,\alpha} = (q^2;q^2)_n (q^{2\alpha+2};q^2)_n, \tag{12}$$

$$(q;q)_{2n+1,\alpha} = (q^2;q^2)_n (q^{2\alpha+2};q^2)_{n+1}. \tag{13}$$

Generalized q-exponential functions

The two Euler's q-analogs of the exponential functions are given by

$$e_q(x) := \sum_{n=0}^{\infty} \frac{x^n}{(q;q)_n} = \frac{1}{(x;q)_\infty} \tag{14}$$

and

$$E_q(x) := \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{(q;q)_n} x^n = (-x;q)_\infty. \tag{15}$$

For $m \geq 1$, we define two generalized q-exponential functions as follows

$$\tilde{E}_{q^m,\alpha}(x) := \sum_{k=0}^{\infty} \frac{q^{mk(k-1)/2}\, x^k}{(q^m;q^m)_{k,\alpha}}, \tag{16}$$

and

$$\tilde{e}_{q^m,\alpha}(x) := \sum_{k=0}^{\infty} \frac{x^k}{(q^m;q^m)_{k,\alpha}}, \quad |x| < 1. \tag{17}$$

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Particular case

Remark that, for $m = 1$ and $\alpha = -\frac{1}{2}$, we have :

$$\tilde{E}_{q,-\frac{1}{2}}(x) = E_q(x), \quad \tilde{e}_{q,-\frac{1}{2}}(x) = e_q(x). \tag{18}$$

Elementary result

For $m = 2$, the following elementary result is useful in the sequel to establish summation formula for gdq-H2P :

$$\tilde{e}_{q^2,-\frac{1}{2}}(x)\tilde{E}_{q^2,-\frac{1}{2}}(y) = \tilde{e}_{q^2,-\frac{1}{2}}(x \oplus_{q^2} y), \tag{19}$$

$$\tilde{e}_{q,-\frac{1}{2}}(x)\tilde{E}_{q^2,-\frac{1}{2}}(-y) = \tilde{e}_q(x \ominus_{q,q^2} y), \;\; \tilde{e}_{q^2,-\frac{1}{2}}(x)\tilde{E}_{q^2,-\frac{1}{2}}(-x) = 1, \tag{20}$$

where

$$(a \ominus_{q,q^2} b)^n := n!_q \sum_{k=0}^{n} \frac{(-1)^k q^{k(k-1)}}{(n-k)!_q \, k!_{q^2}} a^{n-k} b^k, \;\; (a \ominus_{q,q^2} b)^0 := 1.$$

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Generalized discrete q-Hermite II polynomials

Discrete q-Hermite II polynomials

$$\tilde{h}_n(x|q) := (q;q)_n \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k q^{-2nk+k(2k+1)}\, x^{n-2k}}{(q;q)_{n-2k}\,(q^2;q^2)_k}. \quad (21)$$

For $\alpha > -1$, we define a sequence of generalized discrete q-Hermite II polynomials $\{\tilde{h}_{n,\alpha}(x,y|q)\}_{n=0}^{\infty}$ as follows :

Definition

For $x, y \in \mathbb{R}$, a gdq-H2P$\{\tilde{h}_{n,\alpha}(x,y|q)\}_{n=0}^{\infty}$ are defined by :

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and

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Particular cases of gdq-H2H $\tilde{h}_{n,\alpha}(x,y|q)$

1. For $y=1$, we have

$$\tilde{h}_{n,\alpha}(x,1|q)=\tilde{h}_{n,\alpha}(x;q) \tag{24}$$

where $\tilde{h}_{n,\alpha}(x;q)$ is the generalized discrete q-Hermite II polynomial.

2. For $\alpha=-1/2$ and $y=1$, we have

$$\tilde{h}_{n,-1/2}(x,1|q)=\tilde{h}_{n}(x;q). \tag{25}$$

where $\tilde{h}_{n}(x;q)$ is the discrete q-Hermite II polynomial.

3. Indeed since $\lim_{q\to 1}\frac{(q^a;q)_n}{(1-q)^n}=(a)_n$, one readily verifies that

$$\lim_{q\to 1}\frac{\tilde{h}_{n,-\frac{1}{2}}(\sqrt{1-q^2}x,1|q)}{(1-q^2)^{n/2}}=\frac{h_n^{\alpha+\frac{1}{2}}(x)}{2^n} \tag{26}$$

where $h_n^{\alpha+\frac{1}{2}}(x)$ is the Rosenblums generalized Hermite polynomial.

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Generalized discrete q-Hermite II polynomials

Recursion relation

The recursion relation for gdq-H2P $\{\tilde{h}_{n,\alpha}(x,y|q)\}_{n=0}^{\infty}$ holds true.

$$x\tilde{h}_{n,\alpha}(x,y|q) - y\,q^{-2n+1}(1-q^n)\tilde{h}_{n-1,\alpha}(x,y|q) =$$

$$\frac{1-q^{n+1+\theta_n(2\alpha+1)}}{1-q^{n+1}}\tilde{h}_{n+1,\alpha}(x,y|q).$$

Generalized discrete q-Hermite II polynomials

Theorem 1

We have :

$$\lim_{\alpha\to+\infty}\tilde{h}_{2n,\alpha}(x,y|q)=q^{-n(2n-1)}(q;q)_{2n}\,(-y)^n\,S_n\left(x^2y^{-1}q^{-1};q^2\right)\quad(27)$$

and

$$\lim_{\alpha\to+\infty}\tilde{h}_{2n+1,\alpha}(x,y|q)=q^{-n(2n+1)}(q;q)_{2n+1}\,x\,(-y)^n\,S_n\left(x^2y^{-1}q^{-1};q^2\right)\quad(28)$$

where $S_n(x;q)$ are the Stieltjes-Wigert polynomials.

Generalized discrete q-Hermite II polynomials

Lemma

For $\alpha > -1$, the sequence of gdq-H2P $\{\tilde{h}_{n,\alpha}(x,y|q)\}_{n=0}^{\infty}$ can be written in terms of q-Laguerre polynomials $L_n^{(\alpha)}(x;q)$ as

$$\tilde{h}_{2n,\alpha}(x,y|q) = q^{-n(2n-1)} \frac{(q;q)_{2n}}{(q^{2\alpha+2};q^2)_n} (-y)^n L_n^{(\alpha)}\left(x^2 y^{-1} q^{-2\alpha-1};q^2\right) \tag{29}$$

and

$$\tilde{h}_{2n+1,\alpha}(x,y|q) = q^{-n(2n+1)} \frac{(q;q)_{2n+1}}{(q^{2\alpha+2};q^2)_{n+1}} x(-y)^n L_n^{(\alpha+1)}\left(x^2 y^{-1} q^{-2\alpha-1};q^2\right). \tag{30}$$

Generalized discrete q-Hermite II polynomials

Proposition

For $\alpha > -1$, the sequence of gdq-H2P $\{\tilde{h}_{n,\alpha}(x, y|q)\}_{n=0}^{\infty}$ can be written in terms of basic hypergeometric functions as

$$\tilde{h}_{n,\alpha}(x, y|q) = \frac{(q;q)_n}{(q;q)_{n,\alpha}} x^n \, {}_2\phi_1\left(\begin{matrix} q^{-n}, q^{-n-2\alpha} \\ 0 \end{matrix} \,\middle|\, q^2; -\frac{y\, q^{2\alpha+3}}{x^2} \right).$$

Connection formulae for the generalized discrete q-Hermite II polynomials $\{\tilde{h}_{n,\alpha}(x,y|q)\}_{n=0}^{\infty}$

Theorem 2

The sequence of gdq-H2P $\{\tilde{h}_{n,\alpha}(x,y|q)\}_{n=0}^{\infty}$, satisfies the connection formula

$$\tilde{h}_{n,\alpha}(x,\omega|q) = (q;q)_n \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{q^{-2nk+k(2k+1)}\left(-\omega \oplus_{q^2} y\right)^k}{(q^2;q^2)_k\,(q;q)_{n-2k}} \tilde{h}_{n-2k,\alpha}(x,y|q). \tag{31}$$

Proof. Summation formula

To prove the above Theorem 2, we need the following generating function

$$\tilde{e}_{q^2,-\frac{1}{2}}(-yt^2)\tilde{E}_{q,\alpha}(xt)=\sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}\,t^n}{(q;q)_n}\tilde{h}_{n,\alpha}(x,y|q),\quad |yt|<1. \tag{32}$$

Replacing t by $u\oplus_q t$ in the last generating function, we have

$$\tilde{E}_{q,\alpha}\Big[(u\oplus_q t)x\Big]\tilde{e}_{q^2,-\frac{1}{2}}\Big[-y(u\oplus_q t)^2\Big]=\sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}(u\oplus_q t)^n}{(q;q)_n}\tilde{h}_{n,\alpha}(x,y|q) \tag{33}$$

which can be written as

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$$\tilde{E}_{q,\alpha}\Big[(u \oplus_q t)x\Big] = \tilde{E}_{q^2,-\frac{1}{2}}\Big[y(u \oplus_q t)^2\Big] \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}(u \oplus_q t)^n}{(q;q)_n} \tilde{h}_{n,\alpha}(x,y|q). \tag{34}$$

Proof. Summation formula

To prove the above Theorem 2, we need the following generating function

$$\tilde{e}_{q^2,-\frac{1}{2}}(-yt^2)\tilde{E}_{q,\alpha}(xt) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}\, t^n}{(q;q)_n} \tilde{h}_{n,\alpha}(x,y|q), \quad |yt| < 1. \tag{32}$$

Replacing t by $u \oplus_q t$ in the last generating function, we have

$$\tilde{E}_{q,\alpha}\Big[(u \oplus_q t)x\Big]\tilde{e}_{q^2,-\frac{1}{2}}\Big[-y(u \oplus_q t)^2\Big] = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}(u \oplus_q t)^n}{(q;q)_n} \tilde{h}_{n,\alpha}(x,y|q) \tag{33}$$

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Proof. Summation formula

Replacing y by ω and using various identities, we get :

$$\sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}(u \oplus_q t)^n}{(q;q)_n} \tilde{h}_{n,\alpha}(x,\omega|q) =$$

$$\tilde{e}_{q^2,-\frac{1}{2}}\Big[-\omega(u \oplus_q t)^2\Big] \tilde{E}_{q^2,-\frac{1}{2}}\Big[y(u \oplus_q t)^2\Big] \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}(u \oplus_q t)^n}{(q;q)_n} \tilde{h}_{n,\alpha}(x,y|q).$$

The r.h.s of the last expression can be written as

$$\tilde{e}_{q^2,-\frac{1}{2}}\Big[(-\omega \oplus_{q^2} y)(u \oplus_q t)^2\Big] \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}(u \oplus_q t)^n}{(q;q)_n} \tilde{h}_{n,\alpha}(x,y|q) \qquad (35)$$

or

$$\sum_{r=0}^{\infty} \frac{(-\omega \oplus_{q^2} y)^r (u \oplus_q t)^{2r}}{(q^2;q^2)_r} \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}(u \oplus_q t)^n}{(q;q)_n} \tilde{h}_{n,\alpha}(x,y|q). \qquad (36)$$

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Proof. Summation formula

Let us substitute $n+2r=k \implies r \leq \lfloor k/2 \rfloor$ in the last equation, we get :

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{\lfloor n/2 \rfloor}\frac{(q^{\binom{n-2k}{2}})(-\omega \oplus_{q^2} y)^k}{(q^2;q^2)_k\,(q;q)_{n-2k}}\tilde{h}_{n-2k,\alpha}(x,y|q)\right)(u \oplus_q t)^n. \qquad (37)$$

Summarizing the above calculations, we obtain

$$\sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}(u \oplus_q t)^n}{(q;q)_n}\tilde{h}_{n,\alpha}(x,\omega|q) =$$

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Connection formulae for the gdq-H2P $\{\tilde{h}_{n,\alpha}(x, y|q)\}_{n=0}^{\infty}$

Particular cases

Letting :

(i) $y = 0$ in the assertion of Theorem 2, we get the definition of gdq-H2P, i.e.,

$$\tilde{h}_{n,\alpha}(x, \omega|q) = (q; q)_n \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k q^{-2nk+k(2k+1)} x^{n-2k} \omega^k}{(q^2; q^2)_k (q; q)_{n-2k,\alpha}}; \quad (39)$$

(ii) $\omega = 0$ in the assertion of Theorem 2, we get the inversion formula for gdq-H2P

$$x^n = (q; q)_{n,\alpha} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{q^{-2nk+3k^2} y^k}{(q^2; q^2)_k (q; q)_{n-2k}} \tilde{h}_{n-2k,\alpha}(x, y|q). \quad (40)$$

Conclusion

In this work,

(i) we have introduced gdq-H2P $\tilde{h}_{n,\alpha}(x,y|q)$ and derived several properties.

(ii) Also, we have derived implicit summation formula for gdq-H2P $\tilde{h}_{n,\alpha}(x,y|q)$ by using different analytical means on their generating function.

(iii) For $y=1$, the assertion of Theorem 2 can be expressed in terms of generalized discrete q-Hermite II polynomials $\tilde{h}_{n,\alpha}(x;q)$. The assertion of Theorem 2 can be written in terms of discrete q-Hermite II polynomials $\tilde{h}_n(x;q)$ by choosing $y=1$ and $\alpha=-1/2$.

(iv) This process can be extend to summation formula for more generalized forms of q-Hermite polynomials. This study is under way.

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Thank you for attention ! ! !