$(\mathcal{R}, p, q)-$Rogers-Szegö and Hermite polynomials, and induced deformed quantum algebras

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Outline

Motivations

- Deformed quantum algebras, namely the $q-$ deformed algebras [Jimbo M. (1985), Odzijewicz A. (1998), Quesne C. *et. al* (2003)] and their extensions to $(p, q)-$deformed algebras [Burban I.M. and Klimyk A. U. (1994), Chakrabarti, R. and Jagannathan, R. (1991)], continue to attract much attention.
- One of the main reasons is that these topics represent a meeting point of nowadays fast developing areas in mathematics and physics like the theory of quantum orthogonal polynomials and special functions, quantum groups, integrable systems, quantum and conformal field theories, and statistics.
- Indeed, since the work of Jimbo [Jimbo M. (1985)], these fields have known profound interesting developments which can be partially found, for instance, in the books by Chari and Pressley [Chari V. and Pressley A. (1994)], Klimyk and Schumudgen [Klimyk, A., Schmudgen, K. (1997)] , Ismail Moudard [Ismail Moudard, E. H. (2005)] and references therein.

Motivations

- The two-parameter quantum algebra, $Up, q(gl(2))$, was first introduced in reference [Chakrabarti, R. and Jagannathan, R. (1991)] in view to generalize or/and unify a series of $q-$oscillator algebra variants, known in the earlier physics and mathematics literature on the representation theory of single-parameter quantum algebras.
- Then flourish investigations in the same direction, among which the work of Burban and Klimyk [Burban I.M. and Klimyk A. U. (1994)] on representations of two-parameter quantum groups and models of two parameter quantum algebra $U_{p,q}(su_{1,1})$ and $(p, q)-$deformed oscillator algebra. Almost simultaneously, Gelfand *et al.* [Gelfand I. M. *et al.* (1994)] introduced the $(r, s)-$hypergeometric series satisfying a two-parameter difference equation, including $r-$ and $s-$shift operators.
- These new series reproduce the Burban and Klimyk's $P, Q-$hypergeometric functions. The $(p, q)-$deformation rapidly found applications in physics and mathematical physics as described for instance in [Floreanini, R. *et al.* (1993), HMN and Ngompe Nkouankam, E. B. (2007)] .

Motivations

- Upon recalling a technique of constructing explicit realizations of raising and lowering operators that satisfy an algebra akin to the usual harmonic oscillator algebra, through the use of the three-term recurrence relation and the differentiation expression of Hermite polynomials, Galetti [Galetti, D. (2003)] showed that a similar procedure can be carried out in the case of the three-term recursion relation for Rogers-Szegö and Stieltjes-Wigert polynomials and the Jackson $q-$derivative.
- This technique furnished new realizations of the $q-$deformed algebra associated with the $q-$deformed harmonic oscillator, which obeys well known, spread in the literature, commutation relations.
- In the same vein, inspired by the connection between the Rogers-Szegö polynomials and the $q-$oscillator, Jagannathan and Sridhar [Jagannathan R. and Sridhar R. (2010)] defined $(p, q)-$Rogers-Szegö polynomials, proved that they are connected with the $(p, q)-$deformed oscillator associated with the Jagannathan-Srinavasa $(p, q)-$numbers [Jagannathan R., and K. Srinivasa Rao, K.], and proposed a new realization of this algebra.

Motivations

- In a previous paper [HMN. and Bukweli Kyemba, J. D. (2010)], we proposed a theoretical framework for the $(p, q)-$deformed states' generalization, and provided a generalized deformed quantum algebra, inspired by Odzijewicz's work [Odzijewicz, A. (1998)] on a generalization of $q-$deformed states in which the realizations of creation and annihilation operators are given by multiplication by z and the action of a deformed derivative $\partial_{\mathcal{R},p,q}$ on the space of analytic functions defined on a disc.
- This talk aims at giving a new realization of the previous generalized deformed quantum algebras and an explicit definition of the $(\mathcal{R}, p, q)-$Rogers-Szegö polynomials, together with their three-term recursion relation, and the deformed difference equation engendering the creation and annihilation operators.

General framework

- In [J. Math. Phys. **51** 063518 (2010)], we derived the $(\mathcal{R}, p, q)-$numbers which are a generalization of Heine $q-$number

$$[n]_q = \frac{1-q^n}{1-q}, \qquad n = 0, 1, 2, \cdots \tag{1}$$

and Jagannathan-Srinivasa $(p, q)-$numbers ,

$$[n]_{p,q} = \frac{p^n - q^n}{p-q}, \qquad n = 0, 1, 2, \cdots . \tag{2}$$

- Consider p and q, two positive real numbers such that $0 < q < p$, and a given meromorphic function $\mathcal{R}$, defined on $\mathbb{C} \times \mathbb{C}$ by

$$\mathcal{R}(x, y) = \sum_{k,l=-L}^{\infty} r_{kl} x^k y^l \tag{3}$$

with an eventual isolated singularity at the zero, where r_{kl} are complex numbers, $L \in \mathbb{N} \cup \{0\}$, $\mathcal{R}(p^n, q^n) > 0 \ \forall n \in \mathbb{N}$, and $\mathcal{R}(1, 1) = 0$.

General framework

- Denote by $\mathbb{D}_R = \{z \in \mathbb{C} : |z| < R\}$ a complex disc and by $\mathcal{O}(\mathbb{D}_R)$ the set of holomorphic functions defined on $\mathbb{D}_R$. Then, the $(\mathcal{R}, p, q)-$numbers are given by [J. Math. Phys. **51** 063518 (2010)]

$$[n]_{\mathcal{R},p,q} := \mathcal{R}(p^n, q^n), \qquad n = 0, 1, 2, \cdots \tag{4}$$

leading to define $(\mathcal{R}, p, q)-$factorials

$$[n]!_{\mathcal{R},p,q} := \begin{cases} 1 \quad \text{for} \quad n = 0 \\ \mathcal{R}(p, q)\cdots\mathcal{R}(p^n, q^n) \quad \text{for} \quad n \geq 1, \end{cases} \tag{5}$$

and the $(\mathcal{R}, p, q)-$binomial coefficients

$$\left[\begin{array}{c} m \\ n \end{array}\right]_{\mathcal{R},p,q} := \frac{[m]!_{\mathcal{R},p,q}}{[n]!_{\mathcal{R},p,q}[m-n]!_{\mathcal{R},p,q}}, \quad m, n = 0, 1, 2, \cdots; \quad m \geq n \tag{6}$$

satisfying the relation

$$\left[\begin{array}{c} m \\ n \end{array}\right]_{\mathcal{R},p,q} = \left[\begin{array}{c} m \\ m-n \end{array}\right]_{\mathcal{R},p,q}, \quad m, n = 0, 1, 2, \cdots; \quad m \geq n. \tag{7}$$

General framework

- Recall also the following linear operators defined on $\mathcal{O}(\mathbb{D}_R)$ by (see [J. Math. Phys. **51** 063518 (2010)] and references therein for more details):

$$\begin{aligned} Q &: \varphi \longmapsto Q\varphi(z) := \varphi(qz) \\ P &: \varphi \longmapsto P\varphi(z) := \varphi(pz) \\ \partial_{p,q} &: \varphi \longmapsto \partial_{p,q}\varphi(z) := \frac{\varphi(pz) - \varphi(qz)}{z(p-q)}, \end{aligned} \tag{8}$$

and the $(\mathcal{R}, p, q)-$ derivative given by

$$\partial_{\mathcal{R},p,q} := \partial_{p,q}\frac{p-q}{P-Q}\mathcal{R}(P,Q) = \frac{p-q}{pP-qQ}\mathcal{R}(pP,qQ)\partial_{p,q}. \tag{9}$$

General framework

- The quantum algebra associated with the $(\mathcal{R}, p, q)-$deformation is a quantum algebra, $\mathcal{A}_{\mathcal{R},p,q}$, generated by the set of operators $\{1, A, A^\dagger, N\}$ satisfying the following commutation relations:

$$\begin{aligned} AA^\dagger &= [N+1]_{\mathcal{R},p,q}, \qquad & A^\dagger A &= [N]_{\mathcal{R},p,q}. \\ [N,\ A] &= -A, & \left[N,\ A^\dagger\right] &= A^\dagger \end{aligned} \tag{10}$$

with its realization on $\mathcal{O}(\mathbb{D}_R)$ given by

$$A^\dagger \equiv z, \qquad A \equiv \partial_{\mathcal{R},p,q}, \qquad N \equiv z\partial_z, \tag{11}$$

where $\partial_z \equiv \frac{\partial}{\partial z}$ is the usual derivative on $\mathbb{C}$.

Particular cases

1- **Janagathan-Srinivasa deformation** [(2010)]
It corresponds to $\mathcal{R}(x,y) = \frac{x-y}{p-q}$, and $(p,q)-$numbers and $(p,q)-$factorials:

$$[n]_{p,q} = \frac{p^n - q^n}{p-q}, \quad [n]!_{p,q} = \begin{cases} 1 \quad \text{for} \quad n = 0 \\ \frac{((p,q);(p,q))_n}{(p-q)^n} \quad \text{for} \quad n \geq 1, \end{cases} \tag{12}$$

and for nonnegative integers $n,\ m$:

Proposition 1

$$\begin{aligned} [n]_{p,q} &= \sum_{k=0}^{n-1} p^{n-1-k} q^k, \\ [n+m]_{p,q} &= q^m [n]_{p,q} + p^n [m]_{p,q} = p^m [n]_{p,q} + q^n [m]_{p,q}, \\ [-m]_{p,q} &= -q^{-m} p^{-m} [m]_{p,q}, \\ [n-m]_{p,q} &= q^{-m} [n]_{p,q} - q^{-m} p^{n-m} [m]_{p,q} = p^{-m} [n]_{p,q} - q^{n-m} p^{-m} [m]_{p,q}, \\ [n]_{p,q} &= [2]_{p,q} [n-1]_{p,q} - pq [n-2]_{p,q}. \end{aligned} \tag{13}$$

Particular cases

Proposition 2

The $(p, q)-$binomial coefficients

$$\left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} = \frac{((p,q);(p,q))_n}{((p,q);(p,q))_k((p,q);(p,q))_{n-k}}, \quad 0 \leq k \leq n; \quad n \in \mathbb{N}, \tag{14}$$

$((p,q);(p,q))_m := (p-q)(p^2-q^2)\cdots(p^m-q^m),\ m \in \mathbb{N},$ satisfy the identities:

$$\left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} = \left[\begin{array}{c} n \\ n-k \end{array}\right]_{p,q} = p^{k(n-k)}\left[\begin{array}{c} n \\ k \end{array}\right]_{q/p} = p^{k(n-k)}\left[\begin{array}{c} n \\ n-k \end{array}\right]_{q/p}, \tag{15}$$

Particular cases

with

$$\left[\begin{array}{c} n \\ k \end{array}\right]_{q/p} = \frac{(q/p;q/p)_n}{(q/p;q/p)_k(q/p;q/p)_{n-k}},$$

$(q/p;q/p)_n := (1-q/p)(1-q^2/p^2)\cdots(1-q^n/p^n)$; and the $(p,q)-$shifted factorial

$$\begin{aligned} ((a,b);(p,q))_n &\equiv (a-b)(ap-bq)\cdots(ap^{n-1}-bq^{n-1}) \\ &= \sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_{p,q}(-1)^k p^{(n-k)(n-k-1)/2} q^{k(k-1)/2} a^{n-k} b^k. \end{aligned} \quad (16)$$

- Finally, the $(p,q)-$Janagathan-Srinivasa algebra $\mathcal{A}_{p,q}$ satisfies the following commutation relations:

$$\begin{array}{ll} A\,A^\dagger - pA^\dagger A = q^N, & A\,A^\dagger - qA^\dagger A = p^N \\ [N,\; A^\dagger] = A^\dagger, & [N,\; A] = -A. \end{array} \quad (17)$$

Particular cases

2- **Chakrabarty and Jagannathan deformation** [(1991)]

- It is recovered with $\mathcal{R}(x,y) = \frac{1-xy}{(p^{-1}-q)x}$, and the $(\mathcal{R}, p, q)$-numbers and $(\mathcal{R}, p, q)-$factorials are reduced to (p^{-1}, q)-numbers and $(p^{-1}, q)-$factorials:

$$[n]_{p^{-1},q} = \frac{p^{-n} - q^n}{p^{-1} - q},$$

and

$$[n]!_{p^{-1},q} = \begin{cases} 1 \quad \text{for} \quad n = 0 \\ \frac{((p^{-1},q);(p^{-1},q))_n}{(p^{-1}-q)^n} \quad \text{for} \quad n \geq 1. \end{cases} \tag{18}$$

- The deformation properties stem from Janagathan - Srinivasa ones by replacing the parameter p by p^{-1}.

Particular cases

- The $(\mathcal{R}, p, q)-$derivative is then reduced to $(p^{-1}, q)-$derivative:

$$\partial_{\mathcal{R},p,q} := \partial_{p^{-1},q} = \partial_{p,q}\frac{p-q}{P-Q}\frac{1-PQ}{(p^{-1}-q)P} = \frac{1}{(p^{-1}-q)z}(P^{-1}-Q). \quad (19)$$

- Finally, the $(p, q)-$Chakrabarty and Jagannathan algebra $\mathcal{A}_{p^{-1},q}$, generated by $\{1,\ A,\ A^{\dagger},\ N\}$, satisfies the following commutation relations:

$$\begin{array}{ll} A\,A^{\dagger} - p^{-1}A^{\dagger}A = q^{N}, & A\,A^{\dagger} - qA^{\dagger}A = p^{-N} \\ [N,\ A^{\dagger}] = A^{\dagger} & [N,\ A] = -A. \end{array} \quad (20)$$

Particular cases

3- **MNH *et al*, [J. Phys. A: Math. Theor 40 883543 (2007)] generalized $q-$Quesne deformation [(2003)]**

- $\mathcal{R}(x,y) = \frac{xy-1}{(q-p^{-1})y}$, with generalized (p,q)-Quesne numbers and factorials:

$$[n]^Q_{p,q} = \frac{p^n - q^{-n}}{q-p^{-1}}, \quad [n]!^Q_{p,q} = \begin{cases} 1 & \text{for} \quad n=0 \\ \frac{((p,q^{-1});(p,q^{-1}))_n}{(q-p^{-1})^n} & \text{for} \quad n \geq 1, \end{cases} \tag{21}$$

Proposition 3

For n, m are nonnegative integers,

$$[-m]^Q_{p,q} = -p^{-m}q^m[m]^Q_{p,q}, \tag{22}$$

$$[n+m]^Q_{p,q} = q^{-m}[n]^Q_{p,q} + p^n[m]^Q_{p,q} = p^m[n]^Q_{p,q} + q^{-n}[m]^Q_{p,q}, \tag{23}$$

$$[n-m]^Q_{p,q} = q^m[n]^Q_{p,q} - p^{n-m}q^m[m]^Q_{p,q} = p^{-m}[n]^Q_{p,q} + p^{-m}q^{m-n}[m]^Q_{p,q}, \tag{24}$$

$$[n]^Q_{p,q} = \frac{q-p^{-1}}{p-q^{-1}}[2]^Q_{p,q}[n-1]^Q_{p,q} - pq^{-1}[n-2]^Q_{p,q}. \tag{25}$$

Particular cases

Proposition 4

The $(p, q)-$Quesne binomial coefficients

$$\left[\begin{array}{c} n \\ k \end{array}\right]^{Q}_{p,q} = \frac{((p, q^{-1}); (p, q^{-1}))_n}{((p, q^{-1}); (p, q^{-1}))_k((p, q^{-1}); (p, q^{-1}))_{n-k}}, \quad 0 \leq k \leq n; \quad n \in \mathbb{N}, \tag{26}$$

satisfy the following properties

$$\left[\begin{array}{c} n \\ k \end{array}\right]^{Q}_{p,q} = \left[\begin{array}{c} n \\ n-k \end{array}\right]^{Q}_{p,q} = p^{k(n-k)}\left[\begin{array}{c} n \\ k \end{array}\right]_{1/qp} = p^{k(n-k)}\left[\begin{array}{c} n \\ n-k \end{array}\right]_{1/qp}, \tag{27}$$

$$\left[\begin{array}{c} n+1 \\ k \end{array}\right]^{Q}_{p,q} = p^{k}\left[\begin{array}{c} n \\ k \end{array}\right]^{Q}_{p,q} + q^{-n-1+k}\left[\begin{array}{c} n \\ k-1 \end{array}\right]^{Q}_{p,q}, \tag{28}$$

$$\left[\begin{array}{c} n+1 \\ k \end{array}\right]^{Q}_{p,q} = p^{k}\left[\begin{array}{c} n \\ k \end{array}\right]^{Q}_{p,q} + p^{n+1-k}\left[\begin{array}{c} n \\ k-1 \end{array}\right]^{Q}_{p,q} - (p^{n} - q^{-n})\left[\begin{array}{c} n-1 \\ k-1 \end{array}\right]^{Q}_{p,q}. \tag{29}$$

Particular cases

Proof: It stems from the definition and

$$\left[\begin{array}{c} n \\ k \end{array}\right]^{Q}_{p,q} = \left[\begin{array}{c} n \\ k \end{array}\right]_{p,q^{-1}}. \tag{30}$$

□

- Finally, the algebra $\mathcal{A}^{Q}_{p,q}$, generated by $\{1,\ A,\ A^{\dagger},\ N\}$, associated with $(p, q)-$"Quesne" deformation satisfies the following commutation relations:

$$\begin{array}{ll} p^{-1}A\,A^{\dagger} - A^{\dagger}A = q^{-N-1}, & qA\,A^{\dagger} - A^{\dagger}A = p^{N+1} \\ {[N,\ A^{\dagger}]} = A^{\dagger}, & [N,\ A] = -A. \end{array} \tag{31}$$

Particular cases

4- **MNH and Ngompe Nkouankam** $(p, q; \mu, \nu, h)-$**deformation** [J. Phys. A: Math. Theor **40** 12113 (2007)]

- $\mathcal{R}(x, y) = h(p, q)y^{\nu}/x^{\mu}[\frac{xy-1}{(q-p^{-1})y}]$ with $0 < pq < 1$, $p^{\mu} < q^{\nu-1}$, $p > 1$;
- h is a well behaved real and non-negative function of deformation parameters p and q such that $h(p, q) \to 1$ as $(p, q) \to (1, 1)$.
- The $(\mathcal{R}, p, q)-$numbers become $(p, q; \mu, \nu, h)$-numbers defined by

$$[n]^{\mu,\nu}_{p,q,h} = h(p, q)\frac{q^{\nu n}}{p^{\mu n}}\frac{p^n - q^{-n}}{q - p^{-1}}. \quad (32)$$

Proposition 5

The $(p, q; \mu, \nu, h)-$numbers verify the following properties, for $m, n \in \mathbb{N}$:

$$[-m]^{\mu,\nu}_{p,q,h} = -\frac{q^{-2\nu m+m}}{p^{-2\mu m+m}}[m]^{\mu,\nu}_{p,q,h}, \quad (33)$$

Particular cases

$$\begin{aligned} [n+m]^{\mu,\nu}_{p,q,h} &= \frac{q^{\nu m-m}}{p^{\mu m}}[n]^{\mu,\nu}_{p,q,h} + \frac{q^{\nu n}}{p^{\mu n-n}}[m]^{\mu,\nu}_{p,q,h} \\ &= \frac{q^{\nu m}}{p^{\mu m-m}}[n]^{\mu,\nu}_{p,q,h} + \frac{q^{\nu n-n}}{p^{\mu n}}[m]^{\mu,\nu}_{p,q,h}\,, \end{aligned} \tag{34}$$

$$\begin{aligned} [n-m]^{\mu,\nu}_{p,q,h} &= \frac{q^{-\nu m+m}}{p^{-\mu m}}[n]^{\mu,\nu}_{p,q,h} - \frac{q^{\nu(n-2m)+m}}{p^{\mu(n-2m)-n+m}}[m]^{\mu,\nu}_{p,q,h} \\ &= \frac{q^{-\nu m}}{p^{-\mu m+m}}[n]^{\mu,\nu}_{p,q,h} - \frac{q^{\nu(n-2m)-n+m}}{p^{\mu(n-2m)+m}}[m]^{\mu,\nu}_{p,q,h}, \end{aligned} \tag{35}$$

$$[n]^{\mu,\nu}_{p,q,h} = \frac{q-p^{-1}}{p-q^{-1}}\,\frac{q^{-\nu}}{p^{-\mu}}\,\frac{1}{h(p,q)}[2]^{\mu,\nu}_{p,q,h}[n-1]^{\mu,\nu}_{p,q,h} - \frac{q^{2\nu-1}}{p^{2\nu-1}}[n-2]^{\mu,\nu}_{p,q,h}. \tag{36}$$

Proof: This is direct using the definitions, and rewriting

$$[n]^{\mu,\nu}_{p,q,h} = h(p,q)\,\frac{q^{\nu n}}{p^{\mu n}}[n]^{Q}_{p,q,h}. \tag{37}$$

Particular cases

Proposition 6

The $(p, q, \mu, \nu, h)-$ binomial coefficients

$$\left[\begin{array}{c} n \\ k \end{array}\right]^{\mu,\nu}_{p,q,h} := \frac{[n]!^{\mu,\nu}_{p,q,h}}{[k]!^{\mu,\nu}_{p,q,h}[n-k]!^{\mu,\nu}_{p,q,h}} = \frac{q^{\nu k(n-k)}}{p^{\mu k(n-k)}}\left[\begin{array}{c} n \\ k \end{array}\right]^{Q}_{p,q}, \quad 0 \leq k \leq n; \quad n \in \mathbb{N}, \qquad (38)$$

satisfy the properties:

$$\left[\begin{array}{c} n \\ k \end{array}\right]^{\mu,\nu}_{p,q,h} = \left[\begin{array}{c} n \\ n-k \end{array}\right]^{\mu,\nu}_{p,q,h}, \qquad (39)$$

$$\left[\begin{array}{c} n+1 \\ k \end{array}\right]^{\mu,\nu}_{p,q,h} = \frac{q^{\nu k}}{p^{(\mu-1)k}}\left[\begin{array}{c} n \\ k \end{array}\right]^{\mu,\nu}_{p,q,h} + \frac{q^{(\nu-1)(n+1-k)}}{p^{\mu(n+1-k)}}\left[\begin{array}{c} n \\ k-1 \end{array}\right]^{\mu,\nu}_{p,q,h}, \qquad (40)$$

$$\left[\begin{array}{c} n+1 \\ k \end{array}\right]^{\mu,\nu}_{p,q,h} = \frac{q^{\nu k}}{p^{(\mu-1)k}}\left[\begin{array}{c} n \\ k \end{array}\right]^{\mu,\nu}_{p,q,h} + \frac{q^{\nu(n+1-k)}}{p^{(\mu-1)(n+1-k)}}\left[\begin{array}{c} n \\ k-1 \end{array}\right]^{\mu,\nu}_{p,q,h} - (p^n - q^{-n})\frac{q^{\nu n}}{p^{\mu n}}\left[\begin{array}{c} n-1 \\ k-1 \end{array}\right]^{\mu,\nu}_{p,q,h}. \qquad (41)$$

Particular cases

Proof: By a direct computation, and rewriting the generalized factorials as

$$[n]!^{\mu,\nu}_{p,q,h} = h^n(p,q)\frac{q^{n(n+1)/2}}{p^{n(n+1)/2}}[n]!^{Q}_{p,q}. \tag{42}$$

□

- The algebra $\mathcal{A}^{\mu,\nu}_{p,q,h}$, generated by $\{1,\ A,\ A^\dagger,\ N\}$, satisfies the relations:

$$p^{-1}A\,A^\dagger - \frac{q^\nu}{p^\mu}A^\dagger A = h(p,q)\left(\frac{q^{\nu-1}}{p^\mu}\right)^{N+1},$$
$$qA\,A^\dagger - \frac{q^\nu}{p^\mu}A^\dagger A = h(p,q)\left(\frac{q^\nu}{p^{\mu-1}}\right)^{N+1}$$
$$[N,\ A^\dagger] = A^\dagger, \qquad [N,\ A] = -A. \tag{43}$$

Hermite polynomials and harmonic oscillator approach

- The Hermite polynomials are defined as orthogonal polynomials satisfying the three-term recursion and differentiation relations

$$\mathbb{H}_{n+1}(z) = 2z\mathbb{H}_n(z) - 2n\mathbb{H}_{n-1}(z); \quad \frac{d}{dz}\mathbb{H}_n(z) = 2n\mathbb{H}_{n-1}(z), \tag{44}$$

leading to:

$$\mathbb{H}_{n+1}(z) = \left(2z - \frac{d}{dz}\right)\mathbb{H}_n(z) \tag{45}$$

with the introduction of a raising operator [Galetti, D.(2003)]:

$$\hat{a}_+ := 2z - \frac{d}{dz} \tag{46}$$

such that the set of Hermite polynomials can be generated by the application of this operator to the first polynomial $\mathbb{H}_0(z) = 1$, i.e.,

$$\mathbb{H}_n(z) = \hat{a}_+^n \mathbb{H}_0(z). \tag{47}$$

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- The lowering operator $\hat{a}_-$ is accordingly defined as:

$$\hat{a}_-\mathbb{H}_n(z) := \frac{1}{2}\frac{d}{dz}\mathbb{H}_n(z) = n\mathbb{H}_{n-1}(z). \tag{48}$$

- The number operator is

$$\hat{n} = \hat{a}_+\hat{a}_-. \tag{49}$$

- The operators $\hat{a}_-$ and $\hat{a}_+$ satisfy wellkown canonical commutation relations,

$$[\hat{a}_-,\ \hat{a}_+] = 1, \qquad [\hat{n},\ \hat{a}_-] = -\hat{a}_-, \qquad [\hat{n},\ \hat{a}_+] = \hat{a}_+, \tag{50}$$

although they are not the usual creation and annihilation operators of the quantum mechanical harmonic oscillator.

Hermite polynomials and harmonic oscillator approach

- Further, considering the usual Hilbert space, spanned by the vectors $|n\rangle$, generated from the vacuum $|0\rangle$ by the application of the raising operator $\hat{a}_+$, the following relations hold:

$$\begin{aligned} \langle 0|0\rangle &= 1, \\ |n\rangle &= \hat{a}_+^n|0\rangle, \\ \hat{a}_-|0\rangle &= 0. \end{aligned} \tag{51}$$

- In particular, the next expressions are in order:

$$\begin{aligned} \hat{a}_+|n\rangle &= |n+1\rangle, \\ \hat{a}_-|n\rangle &= |n-1\rangle, \\ \langle m|n\rangle &= n!\delta_{mn}. \end{aligned} \tag{52}$$

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- Now, consider the sequence of polynomials [Jagannathan, R and Sridhar, R. (2010)]:

$$\psi_n(z) = \frac{1}{\sqrt{n!}}\mathbf{h}_n(z), \tag{53}$$

where

$$\mathbf{h}_n(z) := (1+z)^n = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) z^k, \tag{54}$$

obeying the recursion relations and the differential equations for polynomials ψ_n,

$$\frac{d}{dz}\psi_n(z) = \sqrt{n}\psi_{n-1}(z), \tag{55}$$

$$(1+z)\psi_n(z) = \sqrt{n+1}\psi_{n+1}(z), \tag{56}$$

$$(1+z)\frac{d}{dz}\psi_n(z) = n\psi_n(z), \tag{57}$$

$$\frac{d}{dz}\left((1+z)\psi_n(z)\right) = (n+1)\psi_n(z). \tag{58}$$

Hermite polynomials and harmonic oscillator approach

- Introduce the creation (or raising), annihilation (or lowering) and number operators,

$$\hat{a}_+ = (1+z), \qquad \hat{a}_- = \frac{d}{dz}, \qquad \hat{n} = (1+z)\frac{d}{dz}, \tag{59}$$

- The set $\{\psi_n(z)\ |n = 0, 1, 2, \cdots\}$ forms a basis for the Bargman-Fock realization of the harmonic oscillator.

Rogers-Szegö polynomials and q-deformed harmonic oscillator

- In the same vein as above [Jagannathan, R and Sridhar, R. (2010)], perform a construction of the creation, annihilation and number operators from the three-term recursion relation and the $q-$difference equation founding the Rogers-Szegö polynomials.
- The Rogers-Szegö polynomials are defined as

$$H_n(z;q) := \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q z^k, \quad n = 0, 1, 2 \cdots \tag{60}$$

satisfying the three-term recursion relation and q-difference equation:

$$H_{n+1}(z;q) = (1+z)H_n(z;q) - z(1-q^n)H_{n-1}(z;q) \tag{61}$$

$$\partial_q H_n(z;q) = [n]_q H_{n-1}(z;q). \tag{62}$$

Rogers-Szegö polynomials and q-deformed harmonic oscillator

- In the limit case $q \to 1$, the Rogers-Szegö polynomial of degree n $(n = 0, 1, 2, \cdots)$ converges to

$$\mathbf{h}_n(z) = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) z^k.$$

- Defining

$$\psi_n(z; q) = \frac{1}{\sqrt{[n]!_q}} H_n(z) = \frac{1}{\sqrt{[n]!_q}} \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q z^k, \quad n = 0, 1, 2 \cdots, \tag{63}$$

we infer

$$\partial_q \psi_n(z; q) = \sqrt{[n]_q} \psi_{n-1}(z; q) \tag{64}$$

with the property that for $n = 0, 1, 2, \cdots$

$$\partial_q^{n+1} \psi_n(z; q) = 0 \quad \text{and} \quad \partial_q^m \psi_n(z; q) \neq 0 \quad \text{for any } m < n + 1. \tag{65}$$

Rogers-Szegö polynomials and q-deformed harmonic oscillator

- The polynomials $\{\psi_n(z;q) \mid n = 0, 1, 2, \cdots\}$ satisfy the following three-term recursion relation and $q-$difference equation:

$$\sqrt{[n+1]_q}\psi_{n+1}(z;q) = (1+z)\psi_n(z;q) - z(1-q)\sqrt{[n]_q}\psi_{n-1}(z;q) \tag{66}$$

$$\left((1+z) - (1-q)z\,\partial_q\right)\psi_n(z;q) = \sqrt{[n+1]_q}\psi_{n+1}(z;q). \tag{67}$$

- It is then natural to define the number operator N, the creation $A^\dagger$and annihilation A operators as:

$$N\psi_n(z;q) = n\psi_n(z;q) \tag{68}$$

$$A^\dagger = 1 + z - (1-q)z\,\partial_q \quad \text{and} \quad A = \partial_q \tag{69}$$

Rogers-Szegö polynomials and q-deformed harmonic oscillator

- The next relations are immediate:

$$\begin{aligned} N\psi_n(z;q) &= n\psi_n(z;q), && (70)\\ A^\dagger\psi_n(z;q) &= \sqrt{[n+1]_q}\psi_{n+1}(z;q), && (71)\\ A\psi_n(z;q) &= \sqrt{[n]_q}\psi_{n-1}(z;q), && (72)\\ A^\dagger A\psi_n(z;q) &= [n]_q\psi_n(z;q) = [N]_q\psi_n(z;q), && (73)\\ AA^\dagger\psi_n(z;q) &= [n+1]_q\psi_n(z;q) = [N+1]_q\psi_n(z;q). && (74) \end{aligned}$$

- This set of polynomials $\{\psi_n(z;q)\,|n=0,1,2,\cdots\}$ provides a basis for a realization of the q-deformed harmonic oscillator algebra given by

$$AA^\dagger - qA^\dagger A = 1, \qquad [N,\ A] = -A, \qquad [N,\ A^\dagger] = A^\dagger \tag{75}$$

All the ingredients are now set to contruct the recursion relation for the $(\mathcal{R}, p, q)-$Rogers-Szegö polynomials and related difference equation, useful to define the creation, annihilation and number operators for a given $(\mathcal{R}, p, q)-$deformed quantum algebra.

$(\mathcal{R}, p, q)-$generalized Rogers-Szegö polynomials and quantum algebras

Theorem 1

If ϕ_i $(i = 1, 2, 3)$ are functions satisfying the following:

$$\phi_i(p, q) \neq 0 \qquad \text{for } i = 1, 2, 3, \tag{76}$$

$$\phi_i(P, Q)z^k = \phi_i^k(p, q)z^k \quad \text{for } \; z \in \mathbb{C}, \; k = 0, 1, 2, \cdots \quad i = 1, 2 \tag{77}$$

and if moreover the following relation between $(\mathcal{R}, p, q)-$binomial coefficients holds

$$\left[\begin{array}{c} n+1 \\ k \end{array}\right]_{\mathcal{R},p,q} = \phi_1^k(p, q)\left[\begin{array}{c} n \\ k \end{array}\right]_{\mathcal{R},p,q} + \phi_2^{n+1-k}(p, q)\left[\begin{array}{c} n \\ k-1 \end{array}\right]_{\mathcal{R},p,q} - \phi_3(p, q)[n]_{\mathcal{R},p,q}\left[\begin{array}{c} n-1 \\ k-1 \end{array}\right]_{\mathcal{R},p,q} \tag{78}$$

for $1 \leq k \leq n$, then the $(\mathcal{R}, p, q)-$Rogers-Szegö polynomials defined as:

$(\mathcal{R}, p, q)-$generalized Rogers-Szegö polynomials and quantum algebras

$$H_n(z;\mathcal{R},p,q) := \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_{\mathcal{R},p,q} z^k, \quad n = 0, 1, 2\cdots \tag{79}$$

satisfy the three-term recursion relation and $(\mathcal{R}, p, q)-$difference equation

$$\begin{aligned} H_{n+1}(z;\mathcal{R},p,q) &= H_n\left(\phi_1(p,q)z : \mathcal{R},p,q\right) + z\phi_2^n(p,q)H_n\left(z\phi_2^{-1}(p,q);\mathcal{R},p,q\right) \\ &\quad -z\phi_3(p,q)[n]_{\mathcal{R},p,q}H_{n-1}\left(z;\mathcal{R},p,q\right) \end{aligned} \tag{80}$$

$$\partial_{\mathcal{R},p,q}H_n(z;\mathcal{R},p,q) = [n]_{\mathcal{R},p,q}H_{n-1}(z;\mathcal{R},p,q). \tag{81}$$

$(\mathcal{R}, p, q)-$generalized Rogers-Szegö polynomials and quantum algebras

- Setting

$$\psi_n(z;\mathcal{R},p,q) = \frac{1}{\sqrt{[n]!_{\mathcal{R},p,q}}} H_n(z;\mathcal{R},p,q), \tag{82}$$

yields the three-term recursion relation and $(\mathcal{R}, p, q)-$difference equation,

$$\Big(\phi_1(P,Q) + z\phi_2^n(p,q)\phi_2^{-1}(P,Q) - z\phi_3(p,q)\partial_{\mathcal{R},p,q}\Big)\psi_n(z;\mathcal{R},p,q) = \sqrt{[n+1]_{\mathcal{R},p,q}}\psi_{n+1}(z;\mathcal{R},p,q), \tag{83}$$

$$\partial_{\mathcal{R},p,q}\psi_n(z;\mathcal{R},p,q) = \sqrt{[n]_{\mathcal{R},p,q}}\psi_{n-1}(z;\mathcal{R},p,q), \tag{84}$$

with the virtue that, for $n = 0, 1, 2, \cdots$,

$$\partial^{n+1}_{\mathcal{R},p,q}\psi_n(z;\mathcal{R},p,q) = 0 \quad \text{and} \quad \partial^{m}_{\mathcal{R},p,q}\psi_n(z;\mathcal{R},p,q) \neq 0 \quad \text{for any } m < n+1. \tag{85}$$

$(\mathcal{R}, p, q)-$generalized Rogers-Szegö polynomials and quantum algebras 35

- The number N, raising $A^\dagger$ and lowering A operators are defined as:

$$N\psi_n(z;\mathcal{R},p,q) := n\psi_n(z;\mathcal{R},p,q), \tag{86}$$

$$A^\dagger := \left(\phi_1(P,Q) + z\phi_2^N(p,q)\phi_2^{-1}(P,Q) - z\phi_3(p,q)\partial_{\mathcal{R},p,q}\right); \tag{87}$$

$$A := \partial_{\mathcal{R},p,q}. \tag{88}$$

From now, Rogers-Szegö polynomials for different known deformations are easily constructable by determining appropriate corresponding functions ϕ_i, $(i = 1, 2, 3)$.

Particulars cases

1- (p, q)-**Rogers-Szegö polynomials in [Jagannathan, R and Sridhar, R. (2010)]**

- They correspond to choosing

$$\phi_1(x, y) = \phi_2(x, y) = \phi(x, y) = x, \quad \phi_3(x, y) = x - y, \quad \phi(P, Q)z^k = \phi_1^k(p, q)z^k$$

.

- The $(p, q)-$Rogers-Szegö polynomials

$$H_n(z; p, q) = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_{p,q} z^k, \quad n = 0, 1, 2, \cdots \tag{89}$$

satisfy the three-term recursion relation

$$H_{n+1}(z; p, q) = H_n(pz; p, q) + zp^n H_n(p^{-1}z; p, q) - z(p^n - q^n)H_{n-1}(z; p, q), \tag{90}$$

Particulars cases

and $(p, q)-$difference equation

$$\partial_{p,q} H_n(z; p, q) = [n]_{p,q} H_{n-1}(z; p, q). \tag{91}$$

- Finally, the set of polynomials

$$\psi_n(z; p, q) = \frac{1}{\sqrt{[n]!_{p,q}}} H_n(z; p, q), \quad n = 0, 1, 2, \cdots \tag{92}$$

forms a basis for a realization of the $(p, q)-$deformed harmonic oscillator and quantum algebra $\mathcal{A}_{p,q}$ with the number N, annihilation A, and creation $A^\dagger$ operators defined as

$$N\psi_n(z; p, q) = n\psi_n(z; p, q), \tag{93}$$

$$A = \partial_{p,q} \quad \text{and} \quad A^\dagger = P + zp^N P^{-1} - z(p - q)\partial_{p,q}. \tag{94}$$

Particulars cases

2- **Rogers-Szegö polynomials from the Chakrabarty-Jagannathan deformation[(1991)]**

- They are obtained from the previous (p, q) deformation by replacing the parameter p by p^{-1} and the operator of dilatation P by P^{-1}.
- Hence, the $(p^{-1}, q)-$Rogers-Szegö polynomials

$$H_n(z; p^{-1}, q) = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_{p^{-1},q} z^k \quad n = 0, 1, 2, \cdots \tag{95}$$

satisfy the three-term recursion relation and $(p^{-1}, q)-$difference equation:

$$\begin{aligned} H_{n+1}(z; p^{-1}, q) &= H_n(p^{-1}z; p^{-1}, q) + zp^{-n}H_n(pz; p^{-1}, q) \\ &\quad -z(p^{-n} - q^n)H_{n-1}(z; p^{-1}, q) \end{aligned} \tag{96}$$

$$\partial_{p^{-1},q} H_n(z; p, q) = [n]_{p^{-1},q} H_{n-1}(z; p, q). \tag{97}$$

Particulars cases

- Finally, the set of polynomials

$$\psi_n(z; p^{-1}, q) = \frac{1}{\sqrt{[n]!_{p^{-1},q}}} H_n(z; p^{-1}, q), \quad n = 0, 1, 2, \cdots \tag{98}$$

forms a basis for a realization of the $(p^{-1}, q)-$deformed harmonic oscillator and quantum algebra $\mathcal{A}_{p^{-1},q}$ with the number N, annihilation A, creation $A^\dagger$, operators formally defined as:

$$N\psi_n(z; p^{-1}, q) = n\psi_n(z; p^{-1}, q), \tag{99}$$

$$A = \partial_{p^{-1},q} \quad \text{and} \quad A^\dagger = P^{-1} + zp^{-N}P - z(p^{-1} - q)\partial_{p^{-1},q}. \tag{100}$$

Particulars cases

3- (p, q)-**Rogers-Szegö polynomials associated with the Quesne deformed quantum algebra** [J. Phys. A: Math. Theor **40** 883543 (2007)]

- The $(p, q)-$Rogers-Szegö polynomials corresponding to the Quesne deformation are deduced from our generalization by choosing

$$\phi_1(x,y) = \phi_2(x,y) = \phi(x,y) = x, \quad \phi_3(x,y) = y - x^{-1}, \phi(P,Q)z^k = \phi_1^k(p,q)z^k$$

.

- The $(p, q)-$Rogers-Szegö polynomials

$$H_n^Q(z;p,q) = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_{p,q}^{Q} z^k \quad n = 0,1,2,\cdots \tag{101}$$

satisfy the three-term recursion relation and $(p, q)-$difference equation:

$$H_{n+1}^Q(z;p,q) = H_n^Q(pz;p,q) + zp^n H_n^Q(p^{-1}z;p,q) - z(p^n - q^{-n})H_{n-1}^Q(z;p,q) \tag{102}$$

$$\partial_{p,q}^Q H_n^Q(z;p,q) = [n]_{p,q}^Q H_{n-1}^Q(z;p,q). \tag{103}$$

Particulars cases

- Thus, the set of polynomials

$$\psi_n^Q(z;p,q) = \frac{1}{\sqrt{[n]!^Q_{p,q}}} H_n^Q(z;p,q), \quad n = 0, 1, 2, \cdots \tag{104}$$

forms a basis for a realization of the $(p,q)-$ Quesne deformed harmonic oscillator and quantum algebra $\mathcal{A}^Q_{p,q}$ with the number N, annihilation A, and creation $A^\dagger$ ops:

$$N\psi_n^Q(z;p,q) = n\psi_n^Q(z;p,q), \tag{105}$$

$$A = \partial^Q_{p,q} \quad \text{and} \quad A^\dagger = P + zp^N P^{-1} - z(q - p^{-1})\partial_{p,q}. \tag{106}$$

Naturally, setting $p = 1$ gives the Rogers-Szegö polynomials associated with the $q-$Quesne deformation [Quesne, C., Penson, K. A., Tkachuk, V. M.. (2003)].

Particulars cases

4- (p, q)-**Rogers-Szegö polynomials from the Quesne deformed quantum algebra** [MNH, Ngompe Nkouankam, E. B., J. Phys. A: Math. Theor **40** 12113 (2007)]

- The (p, q, μ, ν, h)-Rogers-Szegö polynomials are deduced from the general case by setting:

$$\phi_1(x,y) = x^{1-\mu}y^{\nu}, \quad \phi_2(x,y) = x^{-\mu}y^{\nu-1}, \quad \phi_3(x,y) = \frac{y - x^{-1}}{h(p,q)}$$

.

$$h(p,q) \neq 0; \quad \phi_i(p,q) \neq 0,\ i = 1,2,3; \quad \phi_i(P,Q)z^k = \phi_i(p,q)^k z^k,\ i = 1,2.$$

Particulars cases

- The (p, q, μ, ν, h)-Rogers-Szegö polynomials are defined as follows:

$$H_n(z; p, q, \mu, \nu, h) = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_{p,q,h}^{\mu,\nu} z^k, \quad n = 0, 1, 2 \cdots \qquad (107)$$

with the three-term recursion relation and $(p, q, \mu, \nu, h)-$difference equation:

$$\begin{aligned} H_{n+1}(z; p, q, \mu, \nu, h) \quad = \quad & H_n\left(\frac{q^{\nu}}{p^{\mu-1}}z : p, q, \mu, \nu, h\right) \\ & + z\frac{q^{(\nu-1)n}}{p^{\mu n}} H_n\left(\frac{p^{\nu}}{q^{\nu-1}}z; p, q, \mu, \nu, h\right) \\ & - z\frac{q^{\nu n}}{p^{\mu n}}(p^n - q^{-n}) H_{n-1}(z; p, q, \mu, \nu, h) \end{aligned} \qquad (108)$$

$$\partial_{p,q,h}^{\mu,\nu} H_n(z; p, q, \mu, \nu, h) = [n]_{p,q,h}^{\mu,\nu} H_{n-1}(z; p, q, \mu, \nu, h). \qquad (109)$$

Particulars cases

- Finally, the set of polynomials

$$\psi_n(z;p,q,\mu,\nu,h) = \frac{1}{\sqrt{[n]!^{\mu,\nu}_{p,q,h}}} H_n(z;p,q,\mu,\nu,h), \quad n = 0,1,2,\cdots \tag{110}$$

forms a basis for a realization of the $(p,q,\mu,\nu,h)-$deformed algebra $\mathcal{A}_{p,q,\mu,\nu,h}$ with the number N, annihilation A and creation $A^\dagger$ operators formally defined as:

$$N\psi^Q_n(z;p,q,\mu,\nu,h) = n\psi_n(z;p,q,\mu,\nu,h), \tag{111}$$

$$A = \partial^{\mu,\nu}_{p,q,h} \quad \text{and} \quad A^\dagger = \frac{Q^\nu}{P^{\mu-1}} + z\left(\frac{q^{\nu-1}}{p^\mu}\right)^N \frac{P^\mu}{Q^{\nu-1}} - z\frac{(q-p^{-1})}{h(p,q)}\partial^{\mu,\nu}_{p,q,h}. \tag{112}$$

Continuous $(\mathcal{R}, p, q)-$Hermite polynomials

- A peculiar relation from the theory of $q-$deformation is established between Rogers-Szegö H_n and Hermite $\mathbb{H}_n$ polynomials [Ismail Moudard, E. H. (2005), Jagannathan R., and K. Srinivasa Rao, K., Klimyk, A., Schmudgen, K. (1997), Koekoek, R., Swarttouw, R. F., (1998)]:

$$\mathbb{H}_n(\cos\theta; q) = e^{in\,\theta} H_n(e^{-2i\,\theta}; q) = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q e^{i(n-2k)\theta}, \quad n = 0, 1, 2, \cdots . \tag{113}$$

- Further, all the $q-$Hermite polynomials can be explicitly recovered from the initial one, $\mathbb{H}_0(\cos\theta; q) = 1$, using the three-term recursion relation:

$$\mathbb{H}_{n+1}(\cos\theta; q) = 2\cos\theta\mathbb{H}_n(\cos\theta; q) - (1 - q^n)\mathbb{H}_{n-1}(\cos\theta; q) \tag{114}$$

with $\mathbb{H}_{-1}(\cos\theta; q) = 0$.

Continuous $(\mathcal{R}, p, q)-$Hermite polynomials

- Inspired by the above statements, define the $\mathcal{R}, p, q)-$Hermite polynomials through the $\mathcal{R}, p, q)-$Rogers-Szegö polynomials as:

$$\mathbb{H}_n(\cos\theta;\mathcal{R},p,q) = e^{in\,\theta}H_n(e^{-2i\,\theta};\mathcal{R},p,q), \quad n = 0,1,2,\cdots, \tag{115}$$

satisfying the three-term recursion relation:

Proposition 7

$$\begin{aligned} \mathbb{H}_{n+1}(\cos\theta;\mathcal{R},p,q) &= e^{i\,\theta}\phi_1^{\frac{n}{2}}(p,q)\phi_1(P,Q)\mathbb{H}_n(\cos\theta;\mathcal{R},p,q) + e^{-i\,\theta}\phi_2^{\frac{n}{2}}(p,q) \\ &\times \phi_2^{-1}(P,Q)\mathbb{H}_n(\cos\theta;\mathcal{R},p,q) \\ &- \phi_3(p,q)[n]_{\mathcal{R},p,q}\mathbb{H}_{n-1}(\cos\theta;\mathcal{R},p,q). \end{aligned} \tag{116}$$

Particular cases

1- **Continuous $(p, q)-$Hermite polynomials**

- The continuous $(p, q)-$Hermite polynomials were postulated by Jagannathan R., and K. Srinivasa Rao, K. without any further details.
- From the above generalization, they are given by

$$\mathbb{H}_n(\cos\theta; p, q) = e^{in\theta} H_n(e^{-2i\theta}; p, q) = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_{p,q} e^{i(n-2k)\theta}, \quad n = 0, 1, 2, \cdots . \tag{117}$$

satisfying the three-term recursion relation:

$$\begin{aligned} \mathbb{H}_{n+1}(\cos\theta; p, q) &= p^{\frac{n}{2}}(e^{i\theta}P + e^{-i\theta}P^{-1})\mathbb{H}_n(\cos\theta; p, q) \\ &- (p^n - q^n)\mathbb{H}_{n-1}(\cos\theta; p, q), \end{aligned} \tag{118}$$

with $Pe^{i\theta} = p^{-1/2}e^{i\theta}$.

- In the case $p = 1$, this relation turns to be the well-known three-term recursion relation of continuous $q-$Hermite polynomials.

Particular cases

- As matter of illustration, the first three polynomials are, with $\mathbb{H}_{-1}(\cos\theta; p, q) = 0$ and $\mathbb{H}_0(\cos\theta; p, q) = 1$:

$$\begin{aligned}
\mathbb{H}_1(\cos\theta; p, q) &= p^0(e^{i\theta}P + e^{-i\theta}P^{-1} - (p^0 - q^0)0 = e^{i\theta} + e^{-i\theta} = 2\cos\theta \\
&= \begin{bmatrix} 1 \\ 0 \end{bmatrix}_{p,q} e^{i\theta} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{p,q} e^{-i\theta}. \\
\mathbb{H}_2(\cos\theta; p, q) &= p^{\frac{1}{2}}(e^{i\theta}P + e^{-i\theta}P^{-1})(e^{i\theta} + e^{-i\theta}) - p + q \\
&= e^{2i\theta} + e^{-2i\theta} + p + q = 2\cos 2\theta + p + q \\
&= \begin{bmatrix} 2 \\ 0 \end{bmatrix}_{p,q} e^{2i\theta} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}_{p,q} e^{0i\theta} + \begin{bmatrix} 2 \\ 2 \end{bmatrix}_{p,q} e^{-i\theta}.. \\
\mathbb{H}_3(\cos\theta; p, q) &= p(e^{i\theta}P + e^{-i\theta}P^{-1})(e^{2i\theta} + e^{-2i\theta} + p + q) \\
&- (p^2 - q^2)(e^{i\theta} + e^{-i\theta}) \\
&= e^{3i\theta} + e^{-3i\theta} + (p^2 + pq + q^2)(e^{i\theta} + e^{-i\theta}) \\
&= 2\cos 3\theta + 2(p^2 + pq + q^2)\cos\theta \\
&= \begin{bmatrix} 3 \\ 0 \end{bmatrix}_{p,q} e^{3i\theta} + \begin{bmatrix} 3 \\ 1 \end{bmatrix}_{p,q} e^{i\theta} + \begin{bmatrix} 3 \\ 2 \end{bmatrix}_{p,q} e^{-i\theta} + \begin{bmatrix} 3 \\ 3 \end{bmatrix}_{p,q} e^{-3i\theta}.
\end{aligned}$$

Particular cases

2- **Continuous $(p, q)-$Hermite polynomials from the $(p, q)-$ generalized Quesne deformation** [MNH and Ngompe Nkouankam, E. B., J. Phys. A: Math. Theor **40** 883543 (2007)]

- They are defined as

$$\mathbb{H}_n^Q(\cos\theta; p, q) = e^{in\theta} H_n^Q(e^{-2i\theta}; p, q) = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_{p,q}^{Q} e^{i(n-2k)\theta}, \quad n = 0, 1, 2, \cdots . \tag{119}$$

satisfying the three-term recursion relation

$$\begin{aligned} \mathbb{H}_{n+1}^Q(\cos\theta; p, q) &= p^{\frac{n}{2}}(e^{i\theta}P + e^{-i\theta}P^{-1})\mathbb{H}_n^Q(\cos\theta; p, q) \\ &- (p^n - q^{-n})\mathbb{H}_{n-1}^Q(\cos\theta; p, q). \end{aligned} \tag{120}$$

Particular cases

3- **Continuous $(p, q, \mu, \nu, h)-$Hermite polynomials from the generalized Quesne deformation** [MNH, Ngompe Nkouankam, E. B., J. Phys. A: Math. Theor **40** 12113 (2007)]

- The continuous $(p, q, \mu, \nu, h)-$Hermite polynomials are defined by:

$$\begin{aligned}\mathbb{H}_n(\cos\theta; p, q, \mu, \nu, h) &= e^{in\theta} H_n(e^{-2i\theta}; p, q, \mu, \nu, h) \\ &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q,h}^{\mu,\nu} e^{i(n-2k)\theta}, \quad n = 0, 1, 2, \cdots . \quad (121)\end{aligned}$$

satisfying the three-term recursion relation

$$\begin{aligned}\mathbb{H}_{n+1}(\cos\theta; p, q, \mu, \nu, h) &= \frac{q^{\nu\frac{n}{2}}}{p^{(\mu-1)\frac{n}{2}}} \frac{Q^{\nu}}{P^{\mu-1}} \mathbb{H}_n(\cos\theta; p, q, \mu, \nu, h) \\ &+ \frac{q^{(\nu-1)\frac{n}{2}}}{p^{\mu\frac{n}{2}}} \frac{Q^{-(\nu-1)}}{P^{-\mu}} \mathbb{H}_n(\cos\theta; p, q, \mu, \nu, h) \\ &- (p^n - q^{-n}) \frac{q^{\nu n}}{p^{\mu n}} \mathbb{H}_{n-1}(\cos\theta; p, q, \mu, \nu, h). \quad (122)\end{aligned}$$

Concluding remarks

In this talk:

- We have defined and discussed a formalism for constructing $(\mathcal{R}, p, q)-$deformed Rogers-Szegö polynomials generalizing ordinary and known deformed Rogers-Szegö polynomials studied in the literature. Such new polynomials have also been derived for well spread various deformed algebras.
- A full characterization, including the data of three-term recursion relation and difference equation, has been provided.
- By analogy to the technique used to obtain explicit realizations of creation and annihilation operators for $q-$deformed harmonic oscillator as displayed in [Galetti, D.(2003), Jagannathan, R and Sridhar, R. (2010)], we have succeeded in elaborating new realizations of $(\mathcal{R}, p, q)-$deformed quantum algebras.
- Relevant particular cases and examples have been exhibited.
- Finally, the continuous $(\mathcal{R}, p, q)-$Hermite polynomials have been investigated and illustrated in detail.

THANK YOU FOR YOUR ATTENTION