Orthogonal Polynomials and Painlevé equations

Walter Van Assche

Workshop "Introduction to Orthogonal Polynomials and Applications"

Douala (Cameroon), October 11, 2017

Contents

Painlevé equations (discrete and continuous) appear at various places in the theory of orthogonal polynomials:

Contents

Painlevé equations (discrete and continuous) appear at various places in the theory of orthogonal polynomials:

- Discrete Painlevé equations for the recurrence coefficients of orthogonal polynomials

Contents

Painlevé equations (discrete and continuous) appear at various places in the theory of orthogonal polynomials:

- Discrete Painlevé equations for the recurrence coefficients of orthogonal polynomials
- Painlevé differential equations for the recurrence coefficients (Toda flows)

Contents

Painlevé equations (discrete and continuous) appear at various places in the theory of orthogonal polynomials:

- Discrete Painlevé equations for the recurrence coefficients of orthogonal polynomials
- Painlevé differential equations for the recurrence coefficients (Toda flows)
- Rational solutions of Painlevé equations

Contents

Painlevé equations (discrete and continuous) appear at various places in the theory of orthogonal polynomials:

- Discrete Painlevé equations for the recurrence coefficients of orthogonal polynomials
- Painlevé differential equations for the recurrence coefficients (Toda flows)
- Rational solutions of Painlevé equations
- [Special function solutions of Painlevé equations]

Contents

Painlevé equations (discrete and continuous) appear at various places in the theory of orthogonal polynomials:

- Discrete Painlevé equations for the recurrence coefficients of orthogonal polynomials
- Painlevé differential equations for the recurrence coefficients (Toda flows)
- Rational solutions of Painlevé equations
- [Special function solutions of Painlevé equations]
- [Local asymptotics for orthogonal polynomials at critical points]

Contents

Painlevé equations (discrete and continuous) appear at various places in the theory of orthogonal polynomials:

- Discrete Painlevé equations for the recurrence coefficients of orthogonal polynomials
- Painlevé differential equations for the recurrence coefficients (Toda flows)
- Rational solutions of Painlevé equations
- [Special function solutions of Painlevé equations]
- [Local asymptotics for orthogonal polynomials at critical points]

Orthogonal Polynomials and Painlevé Equations
Australian Mathematical Society Lecture Series, Vol. 27, 2018

There are a number of intriguing connections between Painlevé equations and orthogonal polynomials, and this book is one of the first to provide an introduction to these. Researchers in integrable systems and nonlinear equations will find the many explicit examples where Painlevé equations appear in mathematical analysis very useful. Those interested in the asymptotic behavior of orthogonal polynomials will also find the description of Painlevé transcendants and their use for local analysis near certain critical points helpful to their work.Rational solutions and special function solutions of Painlevé equations are worked out in detail, with a survey of recent results and an outline of their close relationship with orthogonal polynomials. Exercises throughout the book help the reader to get to grips with the material.

The author is a leading authority on orthogonal polynomials, giving this work a unique perspective on Painlevé equations.

The Australian Mathematical Society Lecture Series is intended to operate at the frontiers of mathematics itself and of its teaching, and therefore contains both research monographs and textbooks suitable for graduate or undergraduate students.

CAMBRIDGE
UNIVERSITY PRESS
www.cambridge.org

ISBN 978-1-108-44194-0

Van Assche
Orthogonal Polynomials and Painlevé Equations
CAMBRIDGE

Australian Mathematical Society Lecture Series 27

Orthogonal Polynomials and Painlevé Equations

Walter Van Assche

Painlevé equations

Painlevé equations

What: Second order nonlinear differential equations

$$y'' = R(y', y, x), \qquad R \textit{ rational},$$

that have the *Painlevé property*

Painlevé equations

What: Second order nonlinear differential equations

$$y'' = R(y', y, x), \qquad R \text{ rational},$$

that have the *Painlevé property*

What?: The general solution is free from movable branch points

Painlevé equations

What: Second order nonlinear differential equations

$$y'' = R(y', y, x), \qquad R \textit{ rational},$$

that have the *Painlevé property*

What?: The general solution is free from movable branch points

Who: Painlevé et al. found 50 families (up to Möbius transforms), all could be reduced to known equations and six new (= beginning of 20th century) equations

The six Painlevé equations

$$
\begin{aligned}
P_{\mathrm{I}} \qquad & y'' = 6y^2 + x, \\
P_{\mathrm{II}} \qquad & y'' = 2y^3 + xy + \alpha, \\
P_{\mathrm{III}} \qquad & y'' = \frac{(y')^2}{y} - \frac{y'}{x} + \frac{\alpha y^2 + \beta}{x} + \gamma y^3 + \frac{\delta}{y}, \\
P_{\mathrm{IV}} \qquad & y'' = \frac{(y')^2}{2y} + \frac{3}{2}y^3 + 4xy^2 + 2(x^2 - \alpha)y + \frac{\beta}{y}, \\
P_{\mathrm{V}} \qquad & y'' = \left(\frac{1}{2y} + \frac{1}{y-1}\right)(y')^2 - \frac{y'}{x} + \frac{(y-1)^2}{x^2}\left(\alpha y + \frac{\beta}{y}\right) + \frac{\gamma y}{x} \\
& \quad + \frac{\delta y(y+1)}{y-1}, \\
P_{\mathrm{VI}} \qquad & y'' = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x}\right)(y')^2 - \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x}\right)y' \\
& \quad + \frac{y(y-1)(y-x)}{x^2(x-1)^2}\left(\alpha + \frac{\beta x}{y^2} + \frac{\gamma(x-1)}{(y-1)^2} + \frac{\delta x(x-1)}{(y-x)^2}\right),
\end{aligned}
$$

Discrete Painlevé equations

Discrete Painlevé equations

What: second order nonlinear recurrence equations for which the continuous limit is a Painlevé equation

Discrete Painlevé equations

What: second order nonlinear recurrence equations for which the continuous limit is a Painlevé equation

Property: *singularity confinement* (necessary but not sufficient)

Discrete Painlevé equations

What: second order nonlinear recurrence equations for which the continuous limit is a Painlevé equation

Property: *singularity confinement* (necessary but not sufficient)

Quote: anything simpler is linear, anything more complicated becomes hopelessly complicated.

Discrete Painlevé equations

What: second order nonlinear recurrence equations for which the continuous limit is a Painlevé equation

Property: *singularity confinement* (necessary but not sufficient)

Quote: anything simpler is linear, anything more complicated becomes hopelessly complicated.

Classification: By Hidetaka Sakai (2001): based on rational surfaces associated with affine root systems. Describes the space of initial values which parametrizes all the solutions (Kazuo Okamoto, 1979).

Discrete Painlevé equations

What: second order nonlinear recurrence equations for which the continuous limit is a Painlevé equation

Property: *singularity confinement* (necessary but not sufficient)

Quote: anything simpler is linear, anything more complicated becomes hopelessly complicated.

Classification: By Hidetaka Sakai (2001): based on rational surfaces associated with affine root systems. Describes the space of initial values which parametrizes all the solutions (Kazuo Okamoto, 1979).

Geometry: Fine tuning by Kajiwara-Noumi-Yamada (2017): they also include the symmetry, i.e., the group of Bäcklund transformations.

Discrete Painlevé equation: additive

A partial list is

$$\text{d-P}_{\text{I}} \qquad x_{n+1} + x_n + x_{n-1} = \frac{z_n + a(-1)^n}{x_n} + b,$$

$$\text{d-P}_{\text{II}} \qquad x_{n+1} + x_{n-1} = \frac{x_n z_n + a}{1 - x_n^2},$$

$$\text{d-P}_{\text{IV}} \qquad (x_{n+1} + x_n)(x_n + x_{n-1}) = \frac{(x_n^2 - a^2)(x_n^2 - b^2)}{(x_n + z_n)^2 - c^2},$$

$$\text{d-P}_{\text{V}} \qquad \frac{(x_{n+1} + x_n - z_{n+1} - z_n)(x_n + x_{n-1} - z_n - z_{n-1})}{(x_{n+1} + x_n)(x_n + x_{n-1})} = \frac{[(x_n - z_n)^2 - a^2][(x_n - z_n)^2 - b^2]}{(x_n - c^2)(x_n - d^2)},$$

where $z_n = \alpha n + \beta$ and a, b, c, d are constants.

Discrete Painlevé equations: multiplicative

q-P_{III} $$x_{n+1}x_{n-1} = \frac{(x_n - aq_n)(x_n - bq_n)}{(1 - cx_n)(1 - x_n/c)},$$

q-P_V $$(x_{n+1}x_n - 1)(x_nx_{n-1} - 1) = \frac{(x_n - a)(x_n - 1/a)(x_n - b)(x_n - 1/b)}{(1 - cx_nq_n)(1 - x_nq_n/c)},$$

q-P_{VI} $$\frac{(x_nx_{n+1} - q_nq_{n+1})(x_nx_{n-1} - q_nq_{n-1})}{(x_nx_{n+1} - 1)(x_nx_{n-1} - 1)} = \frac{(x_n - aq_n)(x_n - q_n/a)(x_n - bq_n)(x_n - q_n/b)}{(x_n - c)(x_n - 1/c)(x_n - d)(x_n - 1/d)},$$

where $q_n = q_0q^n$ and a, b, c, d are constants.

Discrete Painlevé equations: asymmetric

α-d-P$_{\mathrm{IV}}$

$$(x_n + y_n)(x_{n+1} + y_n) = \frac{(y_n - a)(y_n - b)(y_n - c)(y_n - d)}{(y_n + \gamma - z_n)(y_n - \gamma - z_n)}$$

$$(x_n + y_n)(x_n + y_{n-1}) = \frac{(x_n + a)(x_n + b)(x_n + c)(x_n + d)}{(x_n + \delta - z_{n+1/2})(x_n - \delta - z_{n+1/2})}.$$

Discrete Painlevé equations: asymmetric

α-d-P_{IV}

$$(x_n + y_n)(x_{n+1} + y_n) = \frac{(y_n - a)(y_n - b)(y_n - c)(y_n - d)}{(y_n + \gamma - z_n)(y_n - \gamma - z_n)}$$

$$(x_n + y_n)(x_n + y_{n-1}) = \frac{(x_n + a)(x_n + b)(x_n + c)(x_n + d)}{(x_n + \delta - z_{n+1/2})(x_n - \delta - z_{n+1/2})}.$$

This corresponds to d-P$(E_6^{(1)}/A_2^{(1)})$ where $E_6^{(1)}$ is the surface type and $A_2^{(1)}$ is the symmetry type.

Sakai's classification

$$
\begin{array}{ccccccccccccccc}
E_8^e & & & & & & & & & & & & & & A_1^q \\
\downarrow & & & & & & & & & & & & & \nearrow & \\
E_8^q & \to & E_7^q & \to & E_6^q & \to & D_5^q & \to & A_4^q & \to & (A_2+A_1)^q & \to & (A_1+A_1)^q & \to & A_1^q \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \mid \qquad \downarrow & & \mid \qquad \downarrow & & \\
E_8^d & \to & E_7^d & \to & E_6^d & \to & D_4^c & \to & A_3^c & \to & \mid \quad (2A_1)^c & \to & \mid \qquad A_1^c & & \\
 & & & & & & & & & \searrow & \downarrow \qquad\quad & \searrow & \downarrow \qquad\quad & & \\
 & & & & & & & & & & A_2^c \qquad\quad & \to & A_1^c \qquad\quad & &
\end{array}
$$

Orthogonal polynomials

$$\int p_n(x)p_m(x)w(x)\,dx = \delta_{m,n}, \qquad p_n(x) = \gamma_n x^n + \cdots$$

Orthogonal polynomials

$$\int p_n(x)p_m(x)w(x)\,dx = \delta_{m,n}, \qquad p_n(x) = \gamma_n x^n + \cdots$$

$$\sum_{k=0}^{\infty} p_n(x_k)p_m(x_k)w_k = \delta_{m,n}.$$

Orthogonal polynomials

$$\int p_n(x)p_m(x)w(x)\,dx = \delta_{m,n}, \qquad p_n(x) = \gamma_n x^n + \cdots$$

$$\sum_{k=0}^{\infty} p_n(x_k)p_m(x_k)w_k = \delta_{m,n}.$$

Three term recurrence relation

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x), \qquad \text{orthonormal}$$

$$xP_n(x) = P_{n+1}(x) + b_n P_n(x) + a_n^2 P_{n-1}(x), \qquad \text{monic polynomials}$$

Orthogonal polynomials

$$\int p_n(x)p_m(x)w(x)\,dx = \delta_{m,n}, \qquad p_n(x) = \gamma_n x^n + \cdots$$

$$\sum_{k=0}^{\infty} p_n(x_k)p_m(x_k)w_k = \delta_{m,n}.$$

Three term recurrence relation

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x), \qquad \text{orthonormal}$$

$$xP_n(x) = P_{n+1}(x) + b_n P_n(x) + a_n^2 P_{n-1}(x), \qquad \text{monic polynomials}$$

Orthogonal polynomials on the unit circle (OPUC)

$$\frac{1}{2\pi}\int_0^{2\pi} \varphi_n(z)\overline{\varphi_m(z)}v(\theta)\,d\theta = \delta_{m,n}, \qquad z = e^{i\theta}, \quad \varphi_n(z) = \kappa_n z^n + \cdots$$

Orthogonal polynomials

$$\int p_n(x)p_m(x)w(x)\,dx = \delta_{m,n}, \qquad p_n(x) = \gamma_n x^n + \cdots$$

$$\sum_{k=0}^{\infty} p_n(x_k)p_m(x_k)w_k = \delta_{m,n}.$$

Three term recurrence relation

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_np_n(x) + a_np_{n-1}(x), \qquad \text{orthonormal}$$

$$xP_n(x) = P_{n+1}(x) + b_nP_n(x) + a_n^2P_{n-1}(x), \qquad \text{monic polynomials}$$

Orthogonal polynomials on the unit circle (OPUC)

$$\frac{1}{2\pi}\int_0^{2\pi} \varphi_n(z)\overline{\varphi_m(z)}v(\theta)\,d\theta = \delta_{m,n}, \qquad z = e^{i\theta}, \quad \varphi_n(z) = \kappa_n z^n + \cdots$$

$$z\Phi_n(z) = \Phi_{n+1}(z) + \overline{\alpha_n}\Phi_n^*(z), \qquad \Phi_n^*(z) = z^n\overline{\Phi}_n(1/z).$$

discrete Painlevé I

Let us consider $w(x) = e^{-x^4+tx^2}$ on $(-\infty, \infty)$.

discrete Painlevé I

Let us consider $w(x) = e^{-x^4+tx^2}$ on $(-\infty, \infty)$.

The symmetry $w(-x) = w(x)$ implies $b_n = 0$, hence

$$xp_n(x) = a_{n+1}p_{n+1}(x) + a_n p_{n-1}(x).$$

Let us consider $w(x) = e^{-x^4+tx^2}$ on $(-\infty, \infty)$.

The symmetry $w(-x) = w(x)$ implies $b_n = 0$, hence

$$xp_n(x) = a_{n+1}p_{n+1}(x) + a_n p_{n-1}(x).$$

The structure relation is

$$p_n'(x) = A_n p_{n-1}(x) + C_n p_{n-3}(x).$$

discrete Painlevé I

Let us consider $w(x) = e^{-x^4+tx^2}$ on $(-\infty, \infty)$.

The symmetry $w(-x) = w(x)$ implies $b_n = 0$, hence

$$xp_n(x) = a_{n+1}p_{n+1}(x) + a_n p_{n-1}(x).$$

The structure relation is

$$p_n'(x) = A_n p_{n-1}(x) + C_n p_{n-3}(x).$$

Compatibility between these two relations gives

$$4a_n^2 \left(a_{n+1}^2 + a_n^2 + a_{n-1}^2 - \frac{t}{2} \right) = n$$

discrete Painlevé I

Let us consider $w(x) = e^{-x^4+tx^2}$ on $(-\infty, \infty)$.

The symmetry $w(-x) = w(x)$ implies $b_n = 0$, hence

$$xp_n(x) = a_{n+1}p_{n+1}(x) + a_n p_{n-1}(x).$$

The structure relation is

$$p_n'(x) = A_n p_{n-1}(x) + C_n p_{n-3}(x).$$

Compatibility between these two relations gives

$$4a_n^2 \left(a_{n+1}^2 + a_n^2 + a_{n-1}^2 - \frac{t}{2}\right) = n$$

This is known as **discrete Painlevé I** (d-$\mathrm{P_I}$).

Some history

- Already in old work of Laguerre (1885),

Some history

- Already in old work of Laguerre (1885),
- Also in work of Shohat (1939),

Some history

- Already in old work of Laguerre (1885),
- Also in work of Shohat (1939),
- Independently found by Freud (1976),

Some history

- Already in old work of Laguerre (1885),
- Also in work of Shohat (1939),
- Independently found by Freud (1976),
- Positive solution analyzed by Nevai (1983).

Some history

- Already in old work of Laguerre (1885),
- Also in work of Shohat (1939),
- Independently found by Freud (1976),
- Positive solution analyzed by Nevai (1983).
- Asymptotic expansion by Máté-Nevai-Zaslavsky (1985)

Some history

- Already in old work of Laguerre (1885),
- Also in work of Shohat (1939),
- Independently found by Freud (1976),
- Positive solution analyzed by Nevai (1983).
- Asymptotic expansion by Máté-Nevai-Zaslavsky (1985)
- Recognized as $\mathrm{d}\text{-}\mathrm{P}_{\mathrm{I}}$ by Fokas-Its-Kitaev (1991).

Some history

- Already in old work of Laguerre (1885),
- Also in work of Shohat (1939),
- Independently found by Freud (1976),
- Positive solution analyzed by Nevai (1983).
- Asymptotic expansion by Máté-Nevai-Zaslavsky (1985)
- Recognized as $\mathrm{d\text{-}P_I}$ by Fokas-Its-Kitaev (1991).
- Relation with continuous Painlevé IV (Magnus, 1995)

Asymptotic behavior of a_n^2

Theorem (Freud)

The recurrence coefficients for the weight $w(x) = e^{-x^4+tx^2}$ *satisfy*

$$\lim_{n\to\infty} \frac{a_n}{n^{1/4}} = \frac{1}{\sqrt[4]{12}}.$$

Special solution

Put $x_n = a_n^2$, then (for $t = 0$)

$$x_n(x_{n+1} + x_n + x_{n-1}) = an, \qquad a = 1/4. \tag{1}$$

For orthogonal polynomials we want a solution with $x_0 = 0$ (because $a_0^2 = 0$) and $x_n > 0$ for $n \geq 1$.

Special solution

Put $x_n = a_n^2$, then (for $t = 0$)

$$x_n(x_{n+1} + x_n + x_{n-1}) = an, \qquad a = 1/4. \tag{1}$$

For orthogonal polynomials we want a solution with $x_0 = 0$ (because $a_0^2 = 0$) and $x_n > 0$ for $n \geq 1$.

Theorem (Lew and Quarles, Nevai)

There is a unique solution of (1) for which $x_0 = 0$ and $x_n > 0$ for all $n \geq 1$.

Langmuir lattice or Kac-van Moerbeke equations

General orthogonal polynomials

$$\int_{\mathbb{R}} P_n(x)P_m(x)\, d\mu(x) = 0, \qquad m \neq n.$$

Langmuir lattice or Kac-van Moerbeke equations

General orthogonal polynomials

$$\int_{\mathbb{R}} P_n(x)P_m(x)\,d\mu(x) = 0, \qquad m \neq n.$$

Theorem

Let μ be a symmetric positive measure on $\mathbb{R}$ for which all the moments exist and let μ_t be the measure for which $d\mu_t(x) = e^{tx^2}\,d\mu(x)$, where $t \in \mathbb{R}$ is such that all the moments of μ_t exist. Then the recurrence coefficients of the orthogonal polynomials for μ_t satisfy the differential-difference equations

$$\frac{d}{dt}a_n^2 = a_n^2(a_{n+1}^2 - a_{n-1}^2), \qquad n \geq 1.$$

Painlevé IV

Put $a_n^2 = x_n$, then

$$\begin{aligned} n &= 4x_n(x_{n+1} + x_n + x_{n-1} - t/2), & (2) \\ x_n' &= x_n(x_{n+1} - x_{n-1}) & (3) \end{aligned}$$

Painlevé IV

Put $a_n^2 = x_n$, then

$$\begin{aligned} n &= 4x_n(x_{n+1} + x_n + x_{n-1} - t/2), & (2) \\ x_n' &= x_n(x_{n+1} - x_{n-1}) & (3) \end{aligned}$$

Painlevé IV

Put $a_n^2 = x_n$, then

$$\begin{aligned} n &= 4x_n(x_{n+1} + x_n + x_{n-1} - t/2), && (2) \\ x_n' &= x_n(x_{n+1} - x_{n-1}) && (3) \end{aligned}$$

Eliminate x_{n+1} and x_{n-1} using (2)–(3)

$$x_n'' = \frac{(x_n')^2}{2x_n} + \frac{3x_n^3}{2} - tx_n^2 + x_n\left(\frac{n}{2} + \frac{t^2}{8}\right) - \frac{n^2}{32x_n}$$

Painlevé IV

Put $a_n^2 = x_n$, then

$$n = 4x_n(x_{n+1} + x_n + x_{n-1} - t/2), \quad (2)$$

$$x_n' = x_n(x_{n+1} - x_{n-1}) \quad (3)$$

Eliminate x_{n+1} and x_{n-1} using (2)–(3)

$$x_n'' = \frac{(x_n')^2}{2x_n} + \frac{3x_n^3}{2} - tx_n^2 + x_n\left(\frac{n}{2} + \frac{t^2}{8}\right) - \frac{n^2}{32x_n}$$

This is **Painlevé IV**.

Orthogonal polynomials on the imaginary line

The weight $e^{-x^4+tx^2}$ behaves well on $\mathbb{R}$, but it also tends to 0 along the imaginary axis $i\mathbb{R}$.

Orthogonal polynomials on the imaginary line

The weight $e^{-x^4+tx^2}$ behaves well on $\mathbb{R}$, but it also tends to 0 along the imaginary axis $i\mathbb{R}$.

Orthogonal polynomials $(Q_n)_n$ on the imaginary axis

$$\int_{-i\infty}^{+i\infty} Q_n(x)Q_m(x)e^{-x^4+tx^2}\,dx = 0, \qquad n \neq m,$$

$$xQ_n(x) = Q_{n+1}(x) - b_n Q_{n-1}(x).$$

Orthogonal polynomials on the imaginary line

The weight $e^{-x^4+tx^2}$ behaves well on $\mathbb{R}$, but it also tends to 0 along the imaginary axis $i\mathbb{R}$.

Orthogonal polynomials $(Q_n)_n$ on the imaginary axis

$$\int_{-i\infty}^{+i\infty} Q_n(x)Q_m(x)e^{-x^4+tx^2}\,dx = 0, \qquad n \neq m,$$

$$xQ_n(x) = Q_{n+1}(x) - b_nQ_{n-1}(x).$$

One has $Q_n(x;t) = (-i)^nP_n(ix;-t)$. Hence $b_n(t) = -a_n^2(-t)$.

Orthogonal polynomials on the imaginary line

The weight $e^{-x^4+tx^2}$ behaves well on $\mathbb{R}$, but it also tends to 0 along the imaginary axis $i\mathbb{R}$.

Orthogonal polynomials $(Q_n)_n$ on the imaginary axis

$$\int_{-i\infty}^{+i\infty} Q_n(x)Q_m(x)e^{-x^4+tx^2}\,dx = 0, \qquad n \neq m,$$

$$xQ_n(x) = Q_{n+1}(x) - b_n Q_{n-1}(x).$$

One has $Q_n(x;t) = (-i)^n P_n(ix;-t)$. Hence $b_n(t) = -a_n^2(-t)$.

The $(b_n)_n$ is the unique **negative** solution of $\mathrm{d}\text{-}\mathrm{P}_\mathrm{I}$ with $b_0 = 0$.

Orthogonal polynomials on the cross

We can combine $\mathbb{R}$ and $i\mathbb{R}$ and look for orthogonal polynomials $(R_n)_n$ for which $(n \neq m)$

$$\alpha \int_{-\infty}^{\infty} R_n(x)R_m(x)e^{-x^4+tx^2}\,dx + \beta \int_{-i\infty}^{+i\infty} R_n(x)R_m(x)e^{-x^4+tx^2}\,|dx| = 0,$$

with $\alpha, \beta > 0$.

Orthogonal polynomials on the cross

We can combine $\mathbb{R}$ and $i\mathbb{R}$ and look for orthogonal polynomials $(R_n)_n$ for which $(n \neq m)$

$$\alpha \int_{-\infty}^{\infty} R_n(x)R_m(x)e^{-x^4+tx^2}\,dx + \beta \int_{-i\infty}^{+i\infty} R_n(x)R_m(x)e^{-x^4+tx^2}\,|dx| = 0,$$

with $\alpha, \beta > 0$.

$$xR_n(x) = R_{n+1}(x) - c_n R_{n-1}(x)$$

Orthogonal polynomials on the cross

We can combine $\mathbb{R}$ and $i\mathbb{R}$ and look for orthogonal polynomials $(R_n)_n$ for which $(n \neq m)$

$$\alpha \int_{-\infty}^{\infty} R_n(x)R_m(x)e^{-x^4+tx^2}\,dx + \beta \int_{-i\infty}^{+i\infty} R_n(x)R_m(x)e^{-x^4+tx^2}\,|dx| = 0,$$

with $\alpha, \beta > 0$.

$$xR_n(x) = R_{n+1}(x) - c_n R_{n-1}(x)$$

The $(c_n)_n$ still satisfy $\mathrm{d\text{-}P_I}$, but with initial conditions

$$c_0 = 0, \quad c_1 = \frac{\alpha m_2(t) - \beta m_2(-t)}{\alpha m_0(t) + \beta m_0(-t)},$$

where

$$m_{2k}(t) = \int_{-\infty}^{\infty} x^{2k} e^{-x^4+tx^2}\,dx$$

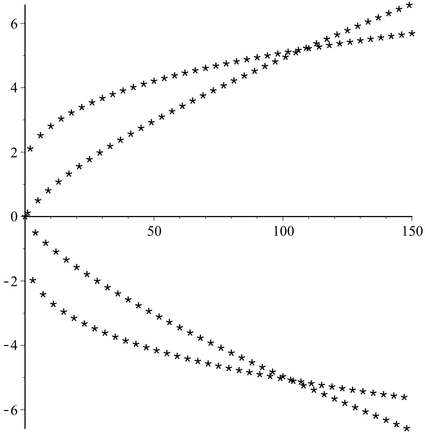
are in terms of parabolic cylinder functions.

Orthogonal polynomials on the cross

The solution of $\mathrm{d}\text{-}\mathrm{P}_{\mathrm{I}}$ behaves very different.

Orthogonal polynomials on the cross

The solution of $\mathrm{d}\text{-}\mathrm{P}_{\mathrm{I}}$ behaves very different. $(t = 0, \beta/\alpha = 1/2)$



Orthogonal polynomials on the cross

$$4x_n \left(x_{n+1} + x_n + x_{n-1} - \frac{t}{2} \right) = n$$

Orthogonal polynomials on the cross

$$4x_n \left(x_{n+1} + x_n + x_{n-1} - \frac{t}{2} \right) = n$$

The case $\alpha = \beta$ and $t = 0$ is special: the initial values are $x_0 = x_1 = 0$.

$$4x_n\left(x_{n+1}+x_n+x_{n-1}-\frac{t}{2}\right)=n$$

The case $\alpha=\beta$ and $t=0$ is special: the initial values are $x_0=x_1=0$.
0 is a singularity of $\mathrm{d}\text{-}\mathrm{P}_\mathrm{I}$ and gives $x_2=\infty$, hence R_3 does not exist.

Orthogonal polynomials on the cross

$$4x_n\left(x_{n+1}+x_n+x_{n-1}-\frac{t}{2}\right)=n$$

The case $\alpha=\beta$ and $t=0$ is special: the initial values are $x_0=x_1=0$.
0 is a singularity of $\mathrm{d}\text{-}\mathrm{P}_{\mathrm{I}}$ and gives $x_2=\infty$, hence R_3 does not exist. The singularity is confined to a finite number of terms.

$$4x_n\left(x_{n+1}+x_n+x_{n-1}-\frac{t}{2}\right)=n$$

The case $\alpha=\beta$ and $t=0$ is special: the initial values are $x_0=x_1=0$.
0 is a singularity of $\mathrm{d}\text{-}\mathrm{P}_{\mathrm{I}}$ and gives $x_2=\infty$, hence R_3 does not exist. The singularity is confined to a finite number of terms.

Property

For $t=0$ and $\alpha=\beta$ one has that R_{4n-1} does not exist for $n\geq 1$.

Orthogonal polynomials on the cross

$$4x_n\left(x_{n+1}+x_n+x_{n-1}-\frac{t}{2}\right)=n$$

The case $\alpha=\beta$ and $t=0$ is special: the initial values are $x_0=x_1=0$.
0 is a singularity of $\mathrm{d}\text{-}\mathrm{P}_{\mathrm{I}}$ and gives $x_2=\infty$, hence R_3 does not exist. The singularity is confined to a finite number of terms.

Property

For $t=0$ and $\alpha=\beta$ one has that R_{4n-1} does not exist for $n\geq 1$. Furthermore

$$R_{4n}(x)=r_n(x^4),\quad R_{4n+1}(x)=xs_n(x^4),\qquad R_{4n+2}(x)=x^2s_n(x^4).$$

discrete Painlevé II

Let $v(\theta) = e^{t \cos \theta}$ on the unit circle

discrete Painlevé II

Let $v(\theta) = e^{t\cos\theta}$ on the unit circle

Trigonometric moments: modified Bessel functions

$$\frac{1}{2\pi}\int_0^{2\pi} e^{in\theta} v(\theta)\, d\theta = I_n(t),$$

discrete Painlevé II

Let $v(\theta) = e^{t\cos\theta}$ on the unit circle

Trigonometric moments: modified Bessel functions

$$\frac{1}{2\pi}\int_0^{2\pi} e^{in\theta} v(\theta)\, d\theta = I_n(t),$$

$v(-\theta) = v(\theta)$ implies that $\alpha_n(t)$ are real valued.

discrete Painlevé II

Let $v(\theta) = e^{t \cos \theta}$ on the unit circle

Trigonometric moments: modified Bessel functions

$$\frac{1}{2\pi} \int_0^{2\pi} e^{in\theta} v(\theta)\, d\theta = I_n(t),$$

$v(-\theta) = v(\theta)$ implies that $\alpha_n(t)$ are real valued.

$$v(\theta) = \hat{v}(z), \quad z = e^{i\theta}, \qquad \hat{v}(z) = \exp\left(t \frac{z + \frac{1}{z}}{2} \right).$$

discrete Painlevé II

Let $v(\theta) = e^{t\cos\theta}$ on the unit circle

Trigonometric moments: modified Bessel functions

$$\frac{1}{2\pi}\int_0^{2\pi} e^{in\theta} v(\theta)\, d\theta = I_n(t),$$

$v(-\theta) = v(\theta)$ implies that $\alpha_n(t)$ are real valued.

$$v(\theta) = \hat{v}(z), \quad z = e^{i\theta}, \qquad \hat{v}(z) = \exp\left(t\frac{z + \frac{1}{z}}{2}\right).$$

Pearson equation

$$\hat{v}'(z) = \frac{t}{2}\left(1 - \frac{1}{z^2}\right)\hat{v}(z).$$

Structure relation

Property

The monic orthogonal polynomials for $v(\theta) = e^{t \cos \theta}$ *satisfy*

$$\Phi_n'(z) = n\Phi_{n-1}(z) + B_n\Phi_{n-2}(z),$$

for a sequence $(B_n)_n$. *In fact, one has*

$$B_n = \frac{t}{2}\frac{\kappa_{n-2}^2}{\kappa_n^2}.$$

Compatibility

$$z\Phi_n(z) = \Phi_{n+1}(z) + \alpha_n \Phi_n^*(x)$$

$$\Phi_n'(z) = n\Phi_{n-1}(z) + B_n \Phi_{n-2}(z)$$

Compatibility

$$z\Phi_n(z) = \Phi_{n+1}(z) + \alpha_n \Phi_n^*(x)$$
$$\Phi_n'(z) = n\Phi_{n-1}(z) + B_n \Phi_{n-2}(z)$$

Theorem (Periwal and Shevitz)

The Verblunsky coefficients for the weight $v(\theta) = e^{t\cos\theta}$ *satisfy*

$$-\frac{t}{2}(\alpha_{n+1} + \alpha_{n-1}) = \frac{(n+1)\alpha_n}{1-\alpha_n^2},$$

with initial values

$$\alpha_{-1} = -1, \quad \alpha_0 = \frac{I_1(t)}{I_0(t)}.$$

Compatibility

$$z\Phi_n(z) = \Phi_{n+1}(z) + \alpha_n \Phi_n^*(x)$$

$$\Phi_n'(z) = n\Phi_{n-1}(z) + B_n \Phi_{n-2}(z)$$

Theorem (Periwal and Shevitz)

The Verblunsky coefficients for the weight $v(\theta) = e^{t\cos\theta}$ satisfy

$$-\frac{t}{2}(\alpha_{n+1} + \alpha_{n-1}) = \frac{(n+1)\alpha_n}{1-\alpha_n^2},$$

with initial values

$$\alpha_{-1} = -1, \quad \alpha_0 = \frac{I_1(t)}{I_0(t)}.$$

This is **discrete Painlevé II** (d-$\mathrm{P_{II}}$)

Discrete Painlevé II

Let $x_n = \alpha_{n-1}$, then

$$x_{n+1} + x_{n-1} = \frac{\alpha n x_n}{1 - x_n^2} \tag{4}$$

Discrete Painlevé II

Let $x_n = \alpha_{n-1}$, then

$$x_{n+1} + x_{n-1} = \frac{\alpha n x_n}{1 - x_n^2} \tag{4}$$

We need a solution with $x_0 = 1$ and $|x_n| < 1$ for $n \geq 1$.

Discrete Painlevé II

Let $x_n = \alpha_{n-1}$, then

$$x_{n+1} + x_{n-1} = \frac{\alpha n x_n}{1 - x_n^2} \tag{4}$$

We need a solution with $x_0 = 1$ and $|x_n| < 1$ for $n \geq 1$.
Such a solution is unique.

Theorem

Suppose $\alpha > 0$. Then there is a unique solution of (4) for which $x_0 = 1$ and $-1 < x_n < 1$.

Discrete Painlevé II

Let $x_n = \alpha_{n-1}$, then

$$x_{n+1} + x_{n-1} = \frac{\alpha n x_n}{1 - x_n^2} \qquad (4)$$

We need a solution with $x_0 = 1$ and $|x_n| < 1$ for $n \geq 1$.
Such a solution is unique.

Theorem

Suppose $\alpha > 0$. Then there is a unique solution of (4) for which $x_0 = 1$ and $-1 < x_n < 1$. The solution corresponds to $x_1 = I_1(2\alpha)/I_0(2/\alpha)$ and is positive for every $n \geq 0$.

Asymptotic behavior

This solution converges to zero (fast) $x_n \to 0$.

Asymptotic behavior

This solution converges to zero (fast) $x_n \to 0$.

Property

The solution of $d\text{-}P_{II}$ *with* $x_0 = 1$ *and* $0 < x_n < 1$ *for* $n \geq 1$ *satisfies*

$$\frac{1}{\alpha^n n! \sum_{k=0}^{n} \frac{\alpha^{-k}}{k!}} \leq x_n \leq \frac{4^n n!}{\alpha^n (2n)!} \sim \frac{1}{\sqrt{2}} \left(\frac{e}{\alpha n}\right)^n .$$

The Ablowitz-Ladik lattice

The lattice equations corresponding to orthogonal polynomials on the unit circle are Ablowitz-Ladik lattice equations (or Schur flow).

The Ablowitz-Ladik lattice

The lattice equations corresponding to orthogonal polynomials on the unit circle are Ablowitz-Ladik lattice equations (or Schur flow).

Theorem

Let ν be a positive measure on the unit circle which is symmetric (the Verblunsky coefficients are real). Let ν_t be the modified measure $d\nu_t(\theta) = e^{t\cos\theta}\, d\nu(\theta)$, with $t \in \mathbb{R}$. The Verblunsky coefficients $(\alpha_n(t))_n$ for the measure ν_t then satisfy

$$2\alpha_n' = (1-\alpha_n^2)(\alpha_{n+1}-\alpha_{n-1}), \qquad n \geq 0.$$

Painlevé V and III

d-$\mathrm{P_{II}}$ gives

$$\alpha_{n+1} + \alpha_{n-1} = \frac{-2n\alpha_n}{t(1-\alpha_n^2)}$$

and Ablowitz-Ladik gives

$$\alpha_{n+1} - \alpha_{n-1} = \frac{2\alpha_n'}{1-\alpha_n^2}.$$

Painlevé V and III

d-P_{II} gives

$$\alpha_{n+1} + \alpha_{n-1} = \frac{-2n\alpha_n}{t(1-\alpha_n^2)}$$

and Ablowitz-Ladik gives

$$\alpha_{n+1} - \alpha_{n-1} = \frac{2\alpha_n'}{1-\alpha_n^2}.$$

Eliminate α_{n+1} and α_{n-1} to find

$$\alpha_n'' = -\frac{\alpha_n}{1-\alpha_n^2}(\alpha_n')^2 - \frac{\alpha_n'}{t} - \alpha_n(1-\alpha_n^2) + \frac{(n+1)^2}{t^2}\frac{\alpha_n}{1-\alpha_n^2}.$$

Painlevé V and III

d-P_{II} gives

$$\alpha_{n+1} + \alpha_{n-1} = \frac{-2n\alpha_n}{t(1-\alpha_n^2)}$$

and Ablowitz-Ladik gives

$$\alpha_{n+1} - \alpha_{n-1} = \frac{2\alpha_n'}{1-\alpha_n^2}.$$

Eliminate α_{n+1} and α_{n-1} to find

$$\alpha_n'' = -\frac{\alpha_n}{1-\alpha_n^2}(\alpha_n')^2 - \frac{\alpha_n'}{t} - \alpha_n(1-\alpha_n^2) + \frac{(n+1)^2}{t^2}\frac{\alpha_n}{1-\alpha_n^2}.$$

If we put $\alpha_n = \frac{1+y}{1-y}$, then y satisfies **Painlevé V** with $\gamma = 0$.

Painlevé V and III

d-P_{II} gives

$$\alpha_{n+1} + \alpha_{n-1} = \frac{-2n\alpha_n}{t(1-\alpha_n^2)}$$

and Ablowitz-Ladik gives

$$\alpha_{n+1} - \alpha_{n-1} = \frac{2\alpha_n'}{1-\alpha_n^2}.$$

Eliminate α_{n+1} and α_{n-1} to find

$$\alpha_n'' = -\frac{\alpha_n}{1-\alpha_n^2}(\alpha_n')^2 - \frac{\alpha_n'}{t} - \alpha_n(1-\alpha_n^2) + \frac{(n+1)^2}{t^2}\frac{\alpha_n}{1-\alpha_n^2}.$$

If we put $\alpha_n = \frac{1+y}{1-y}$, then y satisfies **Painlevé V** with $\gamma = 0$.
The ratio $w_n = \alpha_n/\alpha_{n-1}$ satisfies **Painlevé III** [Hisakado, Tracy-Widom]

Generalized Charlier polynomials

Discrete orthogonal polynomials

$$\sum_{k=0}^{\infty} P_n(k)P_m(k)\frac{c^k}{(\beta)_k k!} = 0, \qquad n \neq m.$$

Generalized Charlier polynomials

Discrete orthogonal polynomials

$$\sum_{k=0}^{\infty} P_n(k)P_m(k)\frac{c^k}{(\beta)_k k!} = 0, \qquad n \neq m.$$

$$\begin{aligned} b_n + b_{n-1} - n + \beta &= \frac{cn}{a_n^2}, \\ (a_{n+1}^2 - c)(a_n^2 - c) &= c(b_n - n)(b_n - n + \beta - 1). \end{aligned}$$

Generalized Charlier polynomials

Discrete orthogonal polynomials

$$\sum_{k=0}^{\infty} P_n(k)P_m(k)\frac{c^k}{(\beta)_k k!} = 0, \qquad n \neq m.$$

$$\begin{aligned} b_n + b_{n-1} - n + \beta &= \frac{cn}{a_n^2}, \\ (a_{n+1}^2 - c)(a_n^2 - c) &= c(b_n - n)(b_n - n + \beta - 1). \end{aligned}$$

This corresponds to a limiting case of **discrete Painlevé with surface/symmetry** $D_4^{(1)}$ in Sakai's classification.

Toda Lattice

Put $c = c_0 e^t$, then

$$\frac{c^k}{(\beta)_k k!} = e^{tk} \frac{c_0^k}{(\beta)_k k!}$$

Toda Lattice

Put $c = c_0 e^t$, then

$$\frac{c^k}{(\beta)_k k!} = e^{tk} \frac{c_0^k}{(\beta)_k k!}$$

Theorem

Suppose μ is a positive measure on the real line and $d\mu_t = e^{tx}\, d\mu(x)$, where t is such that all the moments of μ_t exist. Then the recurrence coefficients $b_n(t)$ and $a_n^2(t)$ of the orthogonal polynomials for the measure μ_t satisfy

$$\begin{aligned} \frac{d}{dt} a_n^2 &= a_n^2 (b_n - b_{n-1}), \qquad n \geq 1, \\ \frac{d}{dt} b_n &= a_{n+1}^2 - a_n^2, \qquad n \geq 0. \end{aligned}$$

generalized Charlier polynomials

Put $x_n = a_n^2$ and $y_n = b_n$ and $x_n' = dx_n/da$, $y_n' = dy_n/da$, then

$$\begin{aligned}(x_n - a)(x_{n+1} - a) &= a(y_n - n)(y_n - n + \beta - 1), \\ y_n + y_{n-1} - n + \beta &= \frac{an}{x_n}\end{aligned}$$

and the Toda lattice equations are

$$\begin{aligned}ax_n' &= x_n(y_n - y_{n-1}), \\ ay_n' &= x_{n+1} - x_n.\end{aligned}$$

generalized Charlier polynomials

Put $x_n = a_n^2$ and $y_n = b_n$ and $x_n' = dx_n/da$, $y_n' = dy_n/da$, then

$$\begin{aligned}(x_n - a)(x_{n+1} - a) &= a(y_n - n)(y_n - n + \beta - 1),\\ y_n + y_{n-1} - n + \beta &= \frac{an}{x_n}\end{aligned}$$

and the Toda lattice equations are

$$\begin{aligned}ax_n' &= x_n(y_n - y_{n-1}),\\ ay_n' &= x_{n+1} - x_n.\end{aligned}$$

Eliminate y_{n-1} and x_{n+1}, and put $x_n = \frac{a}{1-y}$, then $y(a)$ satisfies

$$y'' = \frac{1}{2}\left(\frac{1}{2y} + \frac{1}{y-1}\right)(y')^2 - \frac{y'}{a} + \frac{(1-y)^2}{a^2}\left(\frac{n^2 y}{2} - \frac{(\beta-1)^2}{2y}\right) - \frac{2y}{a}$$

generalized Charlier polynomials

Put $x_n = a_n^2$ and $y_n = b_n$ and $x_n' = dx_n/da$, $y_n' = dy_n/da$, then

$$\begin{aligned} (x_n - a)(x_{n+1} - a) &= a(y_n - n)(y_n - n + \beta - 1), \\ y_n + y_{n-1} - n + \beta &= \frac{an}{x_n} \end{aligned}$$

and the Toda lattice equations are

$$\begin{aligned} ax_n' &= x_n(y_n - y_{n-1}), \\ ay_n' &= x_{n+1} - x_n. \end{aligned}$$

Eliminate y_{n-1} and x_{n+1}, and put $x_n = \frac{a}{1-y}$, then $y(a)$ satisfies

$$y'' = \frac{1}{2}\left(\frac{1}{2y} + \frac{1}{y-1}\right)(y')^2 - \frac{y'}{a} + \frac{(1-y)^2}{a^2}\left(\frac{n^2 y}{2} - \frac{(\beta-1)^2}{2y}\right) - \frac{2y}{a}$$

This is **Painlevé V** with $\delta = 0$.

Put $x_n = a_n^2$ and $y_n = b_n$ and $x_n' = dx_n/da$, $y_n' = dy_n/da$, then

$$\begin{aligned} (x_n - a)(x_{n+1} - a) &= a(y_n - n)(y_n - n + \beta - 1), \\ y_n + y_{n-1} - n + \beta &= \frac{an}{x_n} \end{aligned}$$

and the Toda lattice equations are

$$\begin{aligned} ax_n' &= x_n(y_n - y_{n-1}), \\ ay_n' &= x_{n+1} - x_n. \end{aligned}$$

Eliminate y_{n-1} and x_{n+1}, and put $x_n = \frac{a}{1-y}$, then $y(a)$ satisfies

$$y'' = \frac{1}{2}\left(\frac{1}{2y} + \frac{1}{y-1}\right)(y')^2 - \frac{y'}{a} + \frac{(1-y)^2}{a^2}\left(\frac{n^2 y}{2} - \frac{(\beta-1)^2}{2y}\right) - \frac{2y}{a}$$

This is **Painlevé V** with $\delta = 0$.
This can be transformed to **Painlevé III**.

generalized Meixner polynomials

Discrete orthogonal polynomials

$$\sum_{k=0}^{\infty} P_n(k) P_m(k) \frac{(\gamma)_k a^k}{(\beta)_k k!} = 0, \qquad n \neq m.$$

generalized Meixner polynomials

Discrete orthogonal polynomials

$$\sum_{k=0}^{\infty} P_n(k)P_m(k)\frac{(\gamma)_k a^k}{(\beta)_k k!} = 0, \qquad n \neq m.$$

Put $a_n^2 = na - (\gamma-1)u_n$, and $b_n = n + \gamma - \beta + a - \frac{\gamma-1}{a}v_n$, then

$$\begin{aligned}
(u_n + v_n)(u_{n+1} + v_n) &= \frac{\gamma-1}{a^2} v_n(v_n - a)\left(v_n - a\frac{\gamma-\beta}{\gamma-1}\right), \\
(u_n + v_n)(u_n + v_{n-1}) &= \frac{u_n}{u_n - \frac{an}{\gamma-1}}(u_n + a)\left(u_n + a\frac{\gamma-\beta}{\gamma-1}\right).
\end{aligned}$$

Discrete orthogonal polynomials

$$\sum_{k=0}^{\infty} P_n(k)P_m(k)\frac{(\gamma)_k a^k}{(\beta)_k k!} = 0, \qquad n \neq m.$$

Put $a_n^2 = na - (\gamma-1)u_n$, and $b_n = n+\gamma-\beta+a-\frac{\gamma-1}{a}v_n$, then

$$\begin{aligned}(u_n+v_n)(u_{n+1}+v_n) &= \frac{\gamma-1}{a^2}v_n(v_n-a)\left(v_n - a\frac{\gamma-\beta}{\gamma-1}\right), \\ (u_n+v_n)(u_n+v_{n-1}) &= \frac{u_n}{u_n-\frac{an}{\gamma-1}}(u_n+a)\left(u_n + a\frac{\gamma-\beta}{\gamma-1}\right).\end{aligned}$$

Initial values

$$a_0 = 0, \quad b_0 = \frac{\gamma a}{\beta}\frac{M(\gamma+1,\beta+1,a)}{M(\gamma,\beta,a)}$$

This is **asymmetric discrete Painlevé IV** or $\mathrm{d\text{-}P}(E_6^{(1)}/A_2^{(1)})$.

If we put

$$v_n(a) = \frac{a\Big(ay' - (1+\beta-2\gamma)y^2 + (n+1-a+\beta-2\gamma)y - n\Big)}{2(\gamma-1)(y-1)y},$$

then

$$y'' = \left(\frac{1}{2y} + \frac{1}{y-1}\right)(y')^2 - \frac{y'}{a} + \frac{(y-1)^2}{a^2}\left(Ay + \frac{B}{y}\right) + \frac{Cy}{a} + \frac{Dy(y+1)}{y-1}$$

with

$$A = \frac{(\beta-1)^2}{2}, \quad B = -\frac{n^2}{2}, \quad C = n - \beta + 2\gamma, \quad D = -\frac{1}{2},$$

which is **Painlevé V**.

Other examples

Chen and Its (2010): $w(x) = x^{\alpha} e^{-x} e^{-t/x}$ on $[0, \infty)$

Other examples

Chen and Its (2010): $w(x) = x^\alpha e^{-x} e^{-t/x}$ on $[0,\infty)$
Put $b_n = 2n+\alpha+1+c_n$, $a_n^2 = n(n+\alpha) + y_n + \sum_{j=0}^{n-1} c_j$, and $c_n = 1/x_n$

$$
\begin{aligned}
x_n + x_{n-1} &= \frac{nt - (2n+\alpha)y_n}{y_n(y_n - t)} \\
y_n + y_{n+1} &= t - \frac{2n+\alpha+1}{x_n} - \frac{1}{x_n^2}
\end{aligned}
$$

$\mathrm{d}\text{-}\mathrm{P}((2A_1)^{(1)}/D_6^{(1)})$

Other examples

Chen and Its (2010): $w(x) = x^{\alpha} e^{-x} e^{-t/x}$ on $[0, \infty)$
Put $b_n = 2n + \alpha + 1 + c_n$, $a_n^2 = n(n + \alpha) + y_n + \sum_{j=0}^{n-1} c_j$, and $c_n = 1/x_n$

$$\begin{aligned} x_n + x_{n-1} &= \frac{nt - (2n + \alpha)y_n}{y_n(y_n - t)} \\ y_n + y_{n+1} &= t - \frac{2n + \alpha + 1}{x_n} - \frac{1}{x_n^2} \end{aligned}$$

$\text{d-P}((2A_1)^{(1)}/D_6^{(1)})$

$$c_n'' = \frac{(c_n')^2}{c_n} - \frac{c_n'}{t} + (2n + \alpha + 1)\frac{c_n^2}{t^2} + \frac{c_n^3}{t^2} + \frac{\alpha}{t} - \frac{1}{c_n}$$

which is **Painlevé III**.

Other examples

Basor, Chen, Ehrhardt (2010): $w(x) = (1-x)^\alpha (1+x)^\beta e^{-tx}$

Other examples

Basor, Chen, Ehrhardt (2010): $w(x) = (1-x)^\alpha (1+x)^\beta e^{-tx}$

$$\begin{aligned} tb_n &= 2n+1+\alpha+\beta-t-2R_n, \\ t(t+R_n)a_n^2 &= n(n+\beta) - (2n+\alpha+\beta)r_n - \frac{tr_n(r_n+\alpha)}{R_n} \end{aligned}$$

$$\begin{aligned} 2t(r_{n+1}+r_n) &= 4R_n^2 - 2R_n(2n+1+\alpha+\beta-t) - 2\alpha t, \\ n(n+\beta) - (2n+\alpha+\beta)r_n &= r_n(r_n+\alpha)\left(\frac{t^2}{R_nR_{n-1}} + \frac{t}{R_n} + \frac{t}{R_{n-1}}\right) \end{aligned}$$

Other examples

Basor, Chen, Ehrhardt (2010): $w(x) = (1-x)^\alpha (1+x)^\beta e^{-tx}$

$$\begin{aligned} tb_n &= 2n+1+\alpha+\beta-t-2R_n, \\ t(t+R_n)a_n^2 &= n(n+\beta)-(2n+\alpha+\beta)r_n - \frac{tr_n(r_n+\alpha)}{R_n} \end{aligned}$$

$$\begin{aligned} 2t(r_{n+1}+r_n) &= 4R_n^2 - 2R_n(2n+1+\alpha+\beta-t) - 2\alpha t, \\ n(n+\beta)-(2n+\alpha+\beta)r_n &= r_n(r_n+\alpha)\left(\frac{t^2}{R_nR_{n-1}} + \frac{t}{R_n} + \frac{t}{R_{n-1}}\right) \end{aligned}$$

and for $y = 1 + t/R_n$

$$y'' = \frac{3y-1}{2y(y-1)}(y')^2 - \frac{y'}{t} + 2(2n+1+\alpha+\beta)\frac{y}{t} - \frac{2y(y+1)}{y-1} + \frac{(y-1)^2}{t^2}\left(\frac{\alpha^2 y}{2} - \frac{\beta^2}{2y}\right)$$

which is **Painlevé V**.

Other examples

q-orthogonal polynomials:

$$w(x) = \frac{x^\alpha}{(-x^2;q^2)_\infty(-q^2/x^2;q^2)_\infty}, \qquad x \in [0,\infty)$$

gives rise to **q-discrete Painlevé III**

$$x_{n-1}x_{n+1} = \frac{(x_n + q^{-\alpha})^2}{(q^{n+\alpha}x_n + 1)^2}.$$

Other examples

q-orthogonal polynomials:

$$w(x) = \frac{x^\alpha}{(-x^2;q^2)_\infty(-q^2/x^2;q^2)_\infty}, \qquad x \in [0,\infty)$$

gives rise to **q-discrete Painlevé III**

$$x_{n-1}x_{n+1} = \frac{(x_n + q^{-\alpha})^2}{(q^{n+\alpha}x_n + 1)^2}.$$

$$w(x) = \frac{x^\alpha(-p/x^2;q^2)_\infty}{(-x^2;q^2)_\infty(-q^2/x^2;q^2)_\infty}, \qquad x \in [0,\infty)$$

gives rise to **q-discrete Painlevé V**

$$(z_n z_{n-1} - 1)(z_n z_{n+1} - 1) = \frac{(z_n + \sqrt{q^{2-\alpha}/p})^2(z_n\sqrt{pq^{\alpha-2}})^2}{(q^{n+\alpha/2-1}\sqrt{p}z_n + 1)^2}.$$

Other examples

q-orthogonal polynomials:

$$w(x) = \frac{x^\alpha}{(-x^2;q^2)_\infty(-q^2/x^2;q^2)_\infty}, \qquad x \in [0,\infty)$$

gives rise to **q-discrete Painlevé III**

$$x_{n-1}x_{n+1} = \frac{(x_n + q^{-\alpha})^2}{(q^{n+\alpha}x_n + 1)^2}.$$

$$w(x) = \frac{x^\alpha(-p/x^2;q^2)_\infty}{(-x^2;q^2)_\infty(-q^2/x^2;q^2)_\infty}, \qquad x \in [0,\infty)$$

gives rise to **q-discrete Painlevé V**

$$(z_n z_{n-1} - 1)(z_n z_{n+1} - 1) = \frac{(z_n + \sqrt{q^{2-\alpha}/p})^2(z_n\sqrt{pq^{\alpha-2}})^2}{(q^{n+\alpha/2-1}\sqrt{p}z_n + 1)^2}.$$

$$w(x) = x^\alpha(q^2x^2;q^2)_\infty, \qquad x \in \{q^k, k = 0,1,2,3,\ldots,\}$$

gives again **q-discrete Painlevé V**

Other examples

Bi-orthogonal polynomials on the unit circle (Forrester and Witte, 2006)

$$w(z) = z^{-\mu-\omega}(1+z)^{2\omega_1}(1+tz)^{2\mu} \begin{cases} 1, & \theta \notin (\pi-\phi,\pi), \\ 1-\xi, & \theta \in (\pi-\phi,\pi). \end{cases}, \quad t = e^{i\phi},$$

gives rise to

$$\begin{aligned} g_{n+1}g_n &= t\frac{(f_n+n)(f_n+n+2\mu)}{f_n(f_n-2\omega_1)}, \\ f_n+f_{n-1} &= 2\omega_1 + \frac{n-1+\mu+\omega}{g_n-1} + \frac{(n+\mu+\bar{\omega})t}{g_n-t} \end{aligned}$$

discrete Painlevé V (Sakai's surface $D_4^{(1)}$).

Other examples

Biane (2014) worked out a q-version

$$w(e^{i\theta}) = \left| \frac{(ae^{i\theta}; q)_\infty}{(be^{i\theta}; q)_\infty} \right|^2,$$

and found **discrete Painlevé equations corresponding to $A_3^{(1)}$**

$$r_{n+2}(a-bq^{n+2})+qr_n(a-bq^n) = \frac{2r_{n+1}}{1-r_{n+1}^2}\Big((1-q)(a+bq^{n+1})\beta_{n+1}+(1-q^{n+1})(ab+q)\Big),$$

with

$$\beta_n = \sum_{k=1}^{n} r_r r_{k-1}, \qquad r_n = \Phi_n(0).$$

Other examples

Witte (2015) worked out the most general case (deformed Askey-Wilson polynomials):

Other examples

Witte (2015) worked out the most general case (deformed Askey-Wilson polynomials):
Pearson-type equation for the weight

$$\mathbb{D}_x w(x) = \frac{2V(x)}{W(x)} \mathbb{M}_x w(x),$$

where $\mathbb{D}_x$ is a divided difference operator on a special nonuniform lattice obtained from a quadratic equation

$$Ay^2 + 2Bxy + Cx^2 + 2Dy + 2Ex + F = 0,$$

and $\mathbb{M}_x$ is an average operator for the forward and backward moves on the lattice, and V, W are polynomials.

Other examples

Witte (2015) worked out the most general case (deformed Askey-Wilson polynomials):
Pearson-type equation for the weight

$$\mathbb{D}_x w(x) = \frac{2V(x)}{W(x)} \mathbb{M}_x w(x),$$

where $\mathbb{D}_x$ is a divided difference operator on a special nonuniform lattice obtained from a quadratic equation

$$Ay^2 + 2Bxy + Cx^2 + 2Dy + 2Ex + F = 0,$$

and $\mathbb{M}_x$ is an average operator for the forward and backward moves on the lattice, and V, W are polynomials.

$$W(z) \pm \Delta y V(z) = z^{\mp 3} \prod_{j=1}^{6} (1 - a_j q^{-1/2} z^{\pm 1}),$$

gives the **q-discrete Painlevé cases for $E_7^{(1)}$** in Sakai's scheme.