Orthogonal Polynomials and Random Matrices

Walter Van Assche

Workshop "Introduction to Orthogonal Polynomials and Applications"

Douala (Cameroon), October 8, 2018

Plan of my talks

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talk 1 (Today): **Orthogonal Polynomials and Random Matrices**

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talk 2 (Tuesday am): **Multiple Orthogonal Polynomials**

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talk 3 (Thursday): **Orthogonal Polynomials and Painlevé equations**

Orthogonal polynomials

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Property

$D_n > 0$ *for all* $n \in \mathbb{N}$

Orthogonal polynomials

The monic orthogonal polynomials $P_n(x) = x^n + \cdots$ are given by

$$P_n(x) = \frac{1}{D_n} \det \begin{pmatrix} m_0 & m_1 & m_2 & \cdots & m_n \\ m_1 & m_2 & m_3 & \cdots & m_{n+1} \\ m_2 & m_3 & m_4 & \cdots & m_{n+2} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ m_{n-1} & m_n & m_{n+1} & \cdots & m_{2n-1} \\ 1 & x & x^2 & \cdots & x^n \end{pmatrix}.$$

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$$\frac{1}{\gamma_n^2} = \int_{\mathbb{R}} P_n^2(x) \, d\mu(x) = \frac{D_{n+1}}{D_n}.$$

Vandermonde[1] determinant

Let $x_1, x_2, \ldots, x_n$ be real or complex numbers, then

$$\Delta_n(x_1, \ldots, x_n) = \det \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n \\ x_1^2 & x_2^2 & x_3^2 & \cdots & x_n^2 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ x_1^{n-1} & x_2^{n-1} & x_3^{n-1} & \cdots & x_n^{n-1} \end{pmatrix}.$$

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Property

The Vandermonde determinant can be evaluated and is

$$\Delta_n = \prod_{i>j} (x_i - x_j).$$

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Heine's[2] formula

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The Hankel determinants can be written as an n-fold integral

$$D_n = \frac{1}{n!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Delta_n^2(x_1, \ldots, x_n)\, d\mu(x_1) \cdots d\mu(x_n),$$

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Szegő: [These] Formulas ... are not suitable in general for derivation of properties of the polynomials in question.

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Christoffel-Darboux[4] kernel

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The Christoffel-Darboux formula is

$$\sum_{k=0}^{n-1} \gamma_k^2 P_k(x) P_k(y) = \gamma_{n-1}^2 \frac{P_n(x)P_{n-1}(y) - P_{n-1}(x)P_n(y)}{x-y},$$

and its confluent version

$$\sum_{k=0}^{n-1} \gamma_k^2 P_k^2(x) = \gamma_{n-1}^2 \Big(P_n'(x)P_{n-1}(x) - P_{n-1}'(x)P_n(x) \Big).$$

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$$K_n(x,y) = \sum_{k=0}^{n-1} \gamma_k^2 P_k(x) P_k(y) = \sum_{k=0}^{n-1} p_k(x) p_k(y).$$

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The Christoffel-Darboux kernel is a reproducing kernel: for every polynomial q of degree $\leq n-1$ one has

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$$\int K_n(x,y)K_n(y,z)\,d\mu(y) = K_n(x,z),$$

and

$$\int K_n(x,x)\,d\mu(x) = n.$$

Point process

Definition

We define

$$P(x_1, \dots, x_n) = \frac{1}{n!} \det\big(K_n(x_i, x_j)\big)_{i,j=1}^{n}.$$

This defines the Christoffel-Darboux n-point process in $\mathbb{R}$.

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$P(x_1, \dots, x_n)$ is a probability density on $\mathbb{R}^n$ for the product measure $d\mu(x_1) \cdots d\mu(x_n)$ and one has

$$P(x_1, x_2, \dots, x_n) = \frac{1}{n! D_n} \prod (x_i - x_j)^2 = \frac{1}{n! D_n} \Delta_n^2(x_1, x_2, \dots, x_n).$$

$$\Delta_n(x_1,\ldots,x_n) = \det \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n \\ x_1^2 & x_2^2 & x_3^2 & \cdots & x_n^2 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ x_1^{n-1} & x_2^{n-1} & x_3^{n-1} & \cdots & x_n^{n-1} \end{pmatrix}$$

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$$\rho_k(x_1, \ldots, x_k) = \det\big(K_n(x_i, x_j)\big)_{i,j=1}^{k}.$$

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If $A_i \cap A_j = \emptyset$ $(i \neq j)$, and $N(A)$ is the number of points in A, then

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Property

$$\rho_k(x_1, x_2, \ldots, x_k) = \frac{n!}{(n-k)!} \underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{n-k} P(x_1, \ldots, x_n)\, d\mu(x_{k+1}) \cdots d\mu(x_n).$$

Determinantal point process

Definition

A point process on $\mathbb{R}$ with correlation functions ρ_k is a determinantal point process if there exists a kernel $K(x, y)$ such that for every k and every $x_1, \ldots, x_k \in \mathbb{R}$

$$\rho_k(x_1, x_2, \ldots, x_k) = \det\big(K(x_i, x_j)\big)_{i,j=1}^{k}.$$

Determinantal point process

Theorem

Suppose K is a kernel such that

- $\int K(x,x)\,dx = n \in \mathbb{N}$,
- *For every* $x_1,\ldots,x_n \in \mathbb{R}$, *one has* $\det\big(K(x_i,x_j)\big)_{i,j=1}^k \geq 0$.
- $K(x,y) = \int K(x,s)K(s,y)\,ds$.

Then

$$P(x_1,\ldots,x_n) = \frac{1}{n!}\det\big(K(x_i,x_j)\big)_{i,j=1}^n$$

is a probability density on $\mathbb{R}^n$ which is invariant under permutations of coordinates. The associated n-point process is determinantal.

Determinantal point process

Theorem

Suppose K is a kernel such that

- $\int K(x,x)\,dx = n \in \mathbb{N}$,
- *For every* $x_1,\ldots,x_n \in \mathbb{R}$, *one has* $\det\big(K(x_i,x_j)\big)_{i,j=1}^{k} \geq 0$.
- $K(x,y) = \int K(x,s)K(s,y)\,ds$.

Then

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is a probability density on $\mathbb{R}^n$ *which is invariant under permutations of coordinates. The associated n-point process is determinantal.*

Important example: let $d\mu(x) = w(x)\,dx$, then

$$K(x,y) = K_n(x,y)\sqrt{w(x)}\sqrt{w(y)}.$$

Random matrices

The **Gaussian unitary ensemble** (GUE):
$\mathbf{M}$ is a random $n \times n$ Hermitian matrix with random entries

$$\mathbf{M}_{k,\ell} = X_{k,\ell} + iY_{k,\ell}, \quad \mathbf{M}_{\ell,k} = X_{k,\ell} - iY_{k,\ell}, \qquad k < \ell$$

$$\mathbf{M}_{\mathbf{k},\mathbf{k}} = X_{k,k}, \qquad 1 \leq k \leq n$$

where all $X_{k,\ell}, Y_{k,\ell}, X_{k,k}$ are independent normal random variables with mean zero and variance $\frac{1}{4n}$ (if $k < \ell$) or $\frac{1}{2n}$ (if $k = \ell$).

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$$\frac{1}{Z_n} \prod_{k<\ell} e^{-2n(x_{k,\ell}^2 + y_{k,\ell}^2)} \prod_{k=1}^{n} e^{-nx_{k,k}^2} \prod_{k<\ell} dx_{k,\ell} dy_{k,\ell} \prod_{k=1}^{n} dx_{k,k},$$

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but this is also

$$\frac{1}{Z_n} \exp(-n\mathrm{Tr}\ M^2)\, dM$$

with Z_n a normalizing constant, $M = \left(x_{k,\ell} + iy_{k,\ell}\right)_{1 \leq k\ell, \leq n}$ and $M = M^*$.

Eigenvalues of a random matrix

We are mostly interested in the eigenvalues $\lambda_1, \ldots, \lambda_n$ of the random matrix $\mathbf{M}$.

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Property (Weyl integration formula)

$$dM = c_n \prod_{i<j} (\lambda_i - \lambda_j)^2 \, d\lambda_1 \cdots d\lambda_n \, dU$$

where c_n is a constant and dU is the Haar measure on the unitary group.

[4] Hermann Weyl (1885–1955).

Class functions

Let $\mathcal{H}_n$ be the Hermitian matrices of order n.

Definition

A function $f : \mathcal{H}_n \to \mathbb{C}$ is a class function if

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for all unitary matrices U.

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for all unitary matrices U.

Theorem (Weyl integration formula for class functions)

For an integrable class function f we have

$$\int f(M)\, dM = c_n \int_{\mathbb{R}^n} f(\lambda_1, \ldots, \lambda_n) \prod_{i<j} (\lambda_i - \lambda_j)^2 \, d\lambda_1 \cdots d\lambda_n,$$

with

$$c_n = \frac{\pi^{n(n-1)/2}}{\prod_{j=1}^n j!}.$$

Consequences

For GUE one has

$$\mathbb{E}\det(x\mathbb{I} - \mathbf{M})$$
$$= \frac{1}{n!D_n} \int_{\mathbb{R}^n} \prod_{i=1}^{n} (x - x_i)\ \Delta_n^2(x_1, \ldots, x_n) e^{-n(x_1^2+\cdots+x_n^2)}\, dx_1 \cdots dx_n$$

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The eigenvalues of GUE are a determinantal n-point process with the Christoffel-Darboux kernel for these Hermite polynomials.

The average number of eigenvalues of $\mathbf{M}$ in $[a, b]$ is

$$\int_a^b K_n(x, x)\, e^{-nx^2}\, dx$$

Proof of the Weyl integration formula

For $f : \mathcal{H}_{n+1} \to \mathbb{C}$ and $M \in \mathcal{H}_n$ we define the border transform

$$(B_{n+1}f)(M) = \iiint f \begin{pmatrix} M & (x-iy)^T \\ x+iy & s_0 \end{pmatrix} dx\, dy\, ds_0,$$

where $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$.

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Proof of the Weyl integration formula

3. If $x_j + iy_j = \sqrt{s_j}e^{i\theta_j}$, then for $\lambda = (\lambda_1, \ldots, \lambda_n)$

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$$\mu_{j-1} < \lambda_j < \mu_j, \qquad j = 1, 2, \ldots, n.$$
5. Let $p(t) = \prod_{j=1}^n (t - \lambda_j)$ and $q(t) = \prod_{j=0}^n (t - \mu_j)$, then
$$\frac{q(t)}{p(t)} = t - s_0 - \sum_{j=1}^n \frac{s_j}{t - \lambda_j}.$$

Proof of the Weyl integration formula

6. Given λ with $\lambda_1 < \cdots < \lambda_n$, the transformation $(s_0, s_1, \ldots, s_n) \to \mu = (\mu_0, \ldots, \mu_n)$ is a bijection from $\mathbb{R} \times (\mathbb{R}^+)^n$ onto $D(\lambda) = \{\mu \in \mathbb{R}^{n+1} \mid \mu \text{ interlaces with } \lambda\}$.

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7. The transformation $s \mapsto \mu$ has Jacobian

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$$(B_{n+1}f)(\lambda) = \pi^n \left(\int_{D(\lambda)} f(\mu)\Delta_{n+1}(\mu)\, d\mu \right) \frac{1}{\Delta_n(\lambda)}.$$

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9. Let $D^*(\mu) = \{\lambda \in \mathbb{R}^n \mid \lambda \text{ interlaces with } \mu\}$, then
$$\int_{D^*(\mu)} \Delta_n(\lambda) \, d\lambda = \frac{1}{n!} \Delta_{n+1}(\mu).$$

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Suppose the Weyl integration formula is true for n. If f is a class function on $\mathcal{H}_{n+1}$, then $B_{n+1}f$ is a class function on $\mathcal{H}_n$ and hence

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&= c_n \int (B_{n+1}f)(\lambda)\Delta_n^2(\lambda)\, d\lambda \\
&= c_n n! \int_{\lambda_1 < \cdots < \lambda_n} (B_{n+1}f)(\lambda)\Delta_n^2(\lambda)\, d\lambda.
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$$\int_{\mathcal{H}_{n+1}} f(M)\, dM \quad = \quad c_n n! \pi^n \int_{\lambda_1 < \ldots < \lambda_n} \left(\int_{D(\lambda)} f(\mu) \Delta_{n+1}(\mu)\, d\mu \right) \Delta_n(\lambda)\, d\lambda$$

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$$c_{n+1} = c_n \frac{\pi^n}{(n+1)!}.$$

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Random matrix ensembles

- **Wishart ensemble**: Let **M** be a $n \times m$ matrix with independent complex Gaussian entries $X_{k,\ell} + iY_{k,\ell}$. Then $\mathbf{MM}^*$ has the Wishart distribution with density

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- **Truncated unitary matrices**: Let U be a random unitary matrix of order $(m+k) \times (m+k)$ and let $\mathbf{V}$ be the $m \times n$ upper left corner. Then V^*V is an $n \times n$ matrix and

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References

- P. Deift, **Orthogonal Polynomials and Random Matrices: A Riemann-Hilbert Approach**, Courant Lecture Notes in Mathematics **3**, Courant Institute, New York, NY, and Amer. Math. Soc., Providence, RI., 2009.
- M.E.H. Ismail, **Classical and Quantum Orthogonal Polynomials in One Variable**, Encyclopedia of Mathematics and its Applications 98, Cambridge University Press, 2005 (paperback edition 2009)
- M.L. Mehta, **Random Matrices**, revised and enlarged second edition, Academic Press, San Diego, CA, 1991.
- G. Szegő, **Orthogonal Polynomials**, Amer. Math. Soc. Colloq. Publ. **23**; Providence, RI, 1939; fourth edition 1975.