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Analytic properties of orthogonal polynomials in Sobolev spaces

Francisco Marcellán

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The role of standard orthogonal polynomials

Let μ be a positive Borel measure supported in a subset E of the real line. Determine a polynomial Q_{n-1} of degree $\leq n-1$ that minimizes

$$\int_E |x^n - \pi_{n-1}(x)|^2 d\mu(x).$$

among all the polynomials π_{n-1} of degree $\leq n-1$. This is equivalent to find the n-th monic polynomial orthogonal P_n with respect to the inner product

$$\langle f, g \rangle = \int_E f(x)g(x)d\mu(x). \tag{1}$$

The role of standard orthogonal polynomials

These polynomials satisfy a three term recurrence relation (TTRR)

$$xP_n(x) = P_{n+1}(x) + b_nP_n(x) + a_n^2P_{n-1}(x), n \geq 0, \tag{2}$$

with initial conditions $P_{-1}(x) = 0$ and $P_0(x) = 1$.
The connection with Jacobi matrices and discretization of Schrödinger operators

$$(Ju)_n = a_{n+1}u_{n+1} + b_nu_n + a_nu_{n-1}, n \geq 0, \tag{3}$$

with real entries, $a_n > 0$, for $n \geq 1$, and $a_0 = 0$.

The role of standard orthogonal polynomials

Inverse problem (Favard's Theorem):
Given a sequence of monic polynomials satisfying a TTRR

$$xP_n(x) = P_{n+1}(x) + b_n P_n(x) + a_n^2 P_{n-1}(x), n \geq 0, \tag{4}$$

with initial conditions $P_{-1}(x) = 0$ and $P_0(x) = 1$, they are orthogonal with respect to a positive Borel measure (not unique in general). What about the spectrum of this measure?
This yields the analysis of the spectrum of continuum analog in one dimension of the discrete Schrödinger operator

$$(Hu)(x) = -u''(x) + V(x)u(x) \tag{5}$$

Dense point spectrum vs purely absolutely continuous spectrum.

The role of standard orthogonal polynomials

Zeros of orthogonal polynomials are real, simple and they are located in the interior of the convex hull of the support E of the measure $d\mu$.
Let us denote $x_{n,1} < x_{n,2} < \cdots < x_{n,n}$ the zeros of $P_n(x)$.

Theorem

The zeros of P_n are the eigenvalues of the leading principal submatrix of size $n \times n$ of the Jacobi matrix.

Theorem

(Interlacing property).
Between two consecutive zeros of P_{n+1} there is exactly one zero of P_n.

Zeros of orthogonal polynomials as nodes in Gaussian quadrature and positivity of the Christoffel-Cotes numbers as a consequence of the interlacing property.

The role of standard orthogonal polynomials

Theorem

Given a decreasing sequence $(s_n)_{n\geq 1}$ and an increasing sequence $(r_n)_{n\geq 1}$ of real numbers, there exists a sequence of monic orthogonal polynomials such that $s_n = x_{n,1}$ and $r_n = x_{n,n}$

On the other hand, it is easy to describe the essential support of $d\mu$ in terms of the zeros of $P_n(x)$ for large n.

Theorem

1. *If $y_0 \in E$, then for every $\varepsilon > 0$ $P_n(x)$ has zeros in $(y_0 - \varepsilon, y_0 + \varepsilon)$ for n large enough.*
2. *If $(a, b) \bigcap E = \emptyset$, then for every n, $P_n(x)$ has at most one zero in (a, b).*
3. *If the sequence $(a_n)_{n\geq 1}$ is bounded and y_0 does not belong to E, then there is a $\delta > 0$, so for each n at most one of P_n and P_{n+1} has a zero in $(y_0 - \delta, y_0 + \delta)$.*

The role of standard orthogonal polynomials

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The role of standard orthogonal polynomials

Asymptotics of orthogonal polynomials

1. Outer Strong Asymptotics.

$$\lim_{n\to\infty} \frac{P_n(z)}{\Phi^n(z)}.$$

Szegő class of measures when E is bounded. Techniques of Classical Complex Analysis.

2. Outer Ratio Asymptotics.

$$\lim_{n\to\infty} \frac{P_{n+1}(z)}{P_n(z)}.$$

Nevai class of measures when E is bounded. Techniques of difference equations (Poincaré Theorem)

3. N-th root Asymptotics.

$$\lim_{n\to\infty} |P_n(z)|^{1/n}.$$

Regular class of measures when E is bounded. Techniques of Potential Theory (Green functions)

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Sobolev OP

An inner product is said to be a Sobolev inner product if

$$\langle f, g\rangle_S := \int_{E_0} f(x)g(x)d\mu_0 + \sum_{k=1}^{m} \int_{E_k} f^{(k)}(x)g^{(k)}(x)d\mu_k,$$

where $(d\mu_0, \cdots, d\mu_m)$ is a vector of positive Borel measures and E_k= supp $d\mu_k$, $k = 0, 1, \cdot\cdot, m$.

Using Gram-Schmidt method for the canonical basis $(x^n)_{n\geq 0}$ you get a sequence of monic orthogonal polynomials. Thus, the n-th orthogonal polynomials is an extremal polynomial in terms of the Sobolev norm among all monic polynomials of degree exactly n.

Taking into account $\langle xf, g\rangle_S \neq \langle f, xg\rangle_S$ these polynomials do not satisfy a TTRR. Thus, a basic property of standard orthogonal polynomials is lost. A natural question is to compare analytic properties of these polynomials and the standard ones.

Sobolev OP

A natural framework is the implementation of spectral methods for Boundary Value Problems for elliptic differential operators. Instead of the use of standard orthogonal polynomials when you deal with the Galerkin approximation in the variational problem it seems to be natural to use Sobolev orthogonal polynomials taking into account the matrix problem reduces to a diagonal problem. Unfortunately, there is not a general theory about these families of orthogonal polynomials like in the standard case.

A second interesting problem is related to quadrature formulas involving derivatives. Again, very few results are known about the zeros of Sobolev orthogonal polynomials as nodes in such quadrature rules.

Third, the analysis of Fourier expansions in terms of Sobolev orthogonal polynomials seems to be more accurate than the standard ones in order to increase the speed of convergence as well as to analyze the behavior of the approximation at the end points of the support of the measures, assuming it is a bounded interval.

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The birth time of Sobolev OP

In 1947 D. C. Lewis stated the following problem in the framework of polynomial least square approximations: Let $\alpha_0, \ldots, \alpha_p$ be monotonic, non-decreasing functions defined on $[a, b]$ and let f be a function on $[a, b]$ that satisfies certain regularity conditions. Determine a polynomial P_n of degree $\leq n$ that minimizes

$$\sum_{k=0}^{p} \int_a^b |f^{(k)}(x) - P_n^{(k)}(x)|^2 d\alpha_k(x).$$

Lewis did not use Sobolev orthogonal polynomials and gave a formula for the reminder term of the approximation as an integral of the Peano kernel. The first paper on Sobolev orthogonal polynomials was published by Althammer in 1962, who attributed his motivation to Lewis's paper. These Sobolev OP are orthogonal with respect to the inner product

$$\langle f, g \rangle_S = \int_{-1}^{1} f(x)g(x)dx + \lambda \int_{-1}^{1} f'(x)g'(x)dx, \quad \lambda > 0. \tag{6}$$

The birth time of Sobolev OP

Let $S_n(\cdot;\lambda)$ denote the orthogonal polynomial of degree n with respect to the inner product $\langle\cdot,\cdot\rangle_S$, normalized so that $S_n(1;\lambda) = 1$, and let P_n denote the n-th Legendre polynomial. The following properties hold for $S_n(\cdot;\lambda)$:

1. $\{S_n(\cdot;\lambda)\}_{n\geq 0}$ satisfies a differential equation

$$\lambda S_n''(x;\lambda) - S_n(x;\lambda) = A_n P_{n+1}'(x) + B_n P_{n-1}'(x),$$

   where A_n and B_n are constants that can be given by explicit formulas.

2. $\{S_n(\cdot;\lambda)\}_{n\geq 0}$ satisfies a recursive relation

$$S_n(x;\lambda) - S_{n-2}(x;\lambda) = a_n(P_n(x) - P_{n-2}(x)), \quad n = 1, 2, \ldots.$$

3. $S_n(\cdot;\lambda)$ has n real simple zeros in $(-1,1)$,

The Sobolev-Legendre polynomials were also studied by Gröbner (1967), who established a version of the Rodrigues formula which states that, up to a constant factor c_n,

$$S_n(x;\lambda) = c_n \frac{\partial^n}{1-\lambda\partial^2}\left((x^2-x)^n - \alpha_n(x^2-x)^{n-1}\right)$$

where α_n are real numbers explicitly given in terms of λ and n.

The birth time of Sobolev OP

Althammer also gave an example in which he replaced dx in the second integral in $\langle \cdot, \cdot \rangle_S$ by $w(x)dx$ with $w(x) = 10$ for $-1 \leq x < 0$ and $w(x) = 1$ for $0 \leq x \leq 1$, and made the observation that $S_2(x; \lambda)$ for this new inner product has one real zero outside of $(-1, 1)$.
Brenner considered the inner product

$$\langle f, g \rangle := \int_0^\infty f(x)g(x)e^{-x}dx + \lambda \int_0^\infty f'(x)g'(x)e^{-x}dx, \quad \lambda > 0,$$

and obtained results similar to those of Althammer.

The birth time of Sobolev OP

An important contribution in the early development of the Sobolev orthogonal polynomials was made in 1973 by Schäfke and Wolf who considered a family of inner products

$$\langle f, g \rangle_S = \sum_{j,k=0}^{\infty} \int_a^b f^{(j)}(x) g^{(k)}(x) v_{j,k}(x) w(x) dx, \tag{7}$$

where w and (a, b) are one of the three classical cases, Hermite, Laguerre and Jacobi, and the functions $v_{j,k}$ are polynomials that satisfy $v_{j,k} = v_{k,j}$, $j, k = 0, 1, 2, \ldots$, and permit writing the inner product (7) as

$$\langle f, g \rangle_S = \int_a^b f(x) \mathcal{B} g(x) w(x) dx \quad \text{with} \quad \mathcal{B} g := w^{-1} \sum_{j,k=0}^{\infty} (-1)^j \partial^j (w v_{j,k} \partial^k) g \tag{8}$$

through integration by parts.

The birth time of Sobolev OP

Under further restrictions on $v_{j,k}$, they narrowed down to eight classes of Sobolev orthogonal polynomials, which they call simple generalization of classical orthogonal polynomials.

The primary tool in the early study of Sobolev orthogonal polynomials is integration by parts. Schäfke and Wolf explored when this tool is applicable and outlined potential Sobolev inner products. It is remarkable that their work appeared in such an early stage of the development of Sobolev orthogonal polynomials.

The birth time of Sobolev OP

The study of Sobolev orthogonal polynomials unexpectedly became largely dormant for nearly two decades, from which it reemerged only when a new ingredient, *coherent pairs*, was introduced by Iserles, Koch, Nørsett and Sanz-Serna in 1991 in the framework of Fourier expansions in terms of Legendre-Sobolev orthogonal polynomials.

The comparison with standard Legendre-Fourier expansions reveals the interest of these new expansions despite the fact you have not explicit expressions neither for the polynomials nor the corresponding kernels.

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Coherent pairs of measures and Sobolev OP

The coherent pair introduced in 1991 is defined for the inner product

$$\langle f, g\rangle_\lambda = \int_a^b f(x)g(x)d\mu_0(x) + \lambda \int_a^b f'(x)g'(x)d\mu_1(x), \tag{9}$$

where $-\infty \leq a < b \leq \infty$, μ_0 and μ_1 are positive Borel measures on the real line with finite moments of all orders. Let $P_n(\cdot; d\mu_i)$ denote the monic orthogonal polynomial of degree n with respect to $d\mu_i$, i=0,1.

Definition

The pair $\{d\mu_0, d\mu_1\}$ is called coherent if there exists a sequence of nonzero real numbers $\{a_n\}_{n\geq 1}$ such that

$$P_n(\cdot; d\mu_1) = \frac{P'_{n+1}(\cdot; d\mu_0)}{n+1} + a_n \frac{P'_n(\cdot; d\mu_0)}{n}, \qquad n \geq 1. \tag{10}$$

Coherent pairs of measures and Sobolev OP

Definition (cont.)

If $[a,b]=[-c,c]$ and $d\mu_0$ and $d\mu_1$ are both even, then $\{d\mu_0, d\mu_1\}$ is called a symmetrically coherent pair if

$$P_n(\cdot;d\mu_1)=\frac{P'_{n+1}(\cdot;d\mu_0)}{n+1}+a_n\frac{P'_{n-1}(\cdot;d\mu_0)}{n-1}, \quad n\geq 2. \tag{11}$$

In the case of $d\mu_1=d\mu_0$, we call $d\mu_0$ self-coherent.

Theorem

If $\{d\mu_0, d\mu_1\}$ is a coherent pair, then

$$S_n(x;\lambda)+b_{n-1}(\lambda)S_{n-1}(x;\lambda)=P_n(x;d\mu_0)+\widehat{a}_{n-1}P_{n-1}(x;d\mu_0), \tag{12}$$

where $\widehat{a}_{n-1}=na_n/(n-1)$ *and* $b_{n-1}(\lambda)=\widehat{a}_{n-1}\|P_{n-1}(\cdot;d\mu_0)\|^2_{d\mu_0}/\|S_{n-1}(\cdot;\lambda)\|^2_{\lambda}$.

Coherent pairs of measures and Sobolev OP

Theorem

If $\{\mathcal{U}_0, \mathcal{U}_1\}$ is a coherent pair, then at least one of them has to be classical in the extended sense.

When $\mathcal{U}_0$ and $\mathcal{U}_1$ are positive definite linear functionals associated with measures $d\mu_0$ and $d\mu_1$, these cases are given as follows (Meijer 1997):

Laguerre case

1. $d\mu_0(x) = (x-\xi)x^{\alpha-1}e^{-x}dx$ and $d\mu_1(x) = x^{\alpha}e^{-x}dx$, where if $\xi < 0$ then $\alpha > 0$, and if $\xi = 0$ then $\alpha > -1$.
2. $d\mu_0(x) = x^{\alpha}e^{-x}dx$ and $d\mu_1(x) = \frac{x^{\alpha+1}e^{-x}}{x-\xi}dx + M\delta_{\xi}$, where if $\xi < 0$, $\alpha > -1$ and $M \geq 0$.
3. $d\mu_0(x) = e^{-x}dx + M\delta_0$ and $d\mu_1(x) = e^{-x}dx$, where $M \geq 0$.

Symmetrically coherent pairs

Jacobi case

1. $d\mu_0(x) = |x-\xi|(1-x)^{\alpha-1}(1+x)^{\beta-1}dx$ and $d\mu_1(x) = (1-x)^{\alpha}(1+x)^{\beta}dx$, where if $|\xi| > 1$ then $\alpha > 0$ and $\beta > 0$, if $\xi = 1$ then $\alpha > -1$ and $\beta > 0$, and if $\xi = -1$ then $\alpha > 0$ and $\beta > -1$.
2. $d\mu_0(x) = (1-x)^{\alpha}(1+x)^{\beta}dx$ and $d\mu_1(x) = \frac{1}{|x-\xi|}(1-x)^{\alpha+1}(1+x)^{\beta+1}dx + M\delta_{\xi}$, where $|\xi| > 1$, $\alpha > -1$ and $\beta > -1$, and $M \geq 0$.
3. $d\mu_0(x) = (1+x)^{\beta-1}dx + M\delta_1$ and $d\mu_1(x) = (1+x)^{\beta}dx$, where $\beta > 0$ and $M \geq 0$.
4. $d\mu_0(x) = (1-x)^{\alpha-1}dx + M\delta_{-1}$ and $d\mu_1(x) = (1-x)^{\alpha}dx$, where $\alpha > 0$ and $M \geq 0$.

The similar analysis was also carried out for symmetrically coherent pairs. It lead to the following list of symmetrically coherent pairs.

Hermite case

1. $d\mu_0(x) = e^{-x^2}dx$ and $d\mu_1(x) = \frac{1}{x^2+\xi^2}e^{-x^2}dx$, where $\xi \neq 0$.
2. $d\mu_0(x) = (x^2+\xi^2)e^{-x^2}dx$ and $d\mu_1(x) = e^{-x^2}dx$, where $\xi \neq 0$.

Symmetrically coherent pairs

Gegenbauer case

1. $d\mu_0(x) = (1-x^2)^{\alpha-1}dx$ and $d\mu_1(x) = \frac{1}{x^2+\xi^2}(1-x^2)^{\alpha}dx$, where $\xi \neq 0$ and $\alpha > 0$.
2. $d\mu_0(x) = (1-x^2)^{\alpha-1}dx$ and $d\mu_1(x) = \frac{1}{\xi^2-x^2}(1-x^2)^{\alpha}dx + M\delta_{\xi} + M\delta_{-\xi}$, where $|\xi| \geq 1$, $\alpha > 0$ and $M \geq 0$.
3. $d\mu_0(x) = (x^2+\xi^2)(1-x^2)^{\alpha-1}dx$ and $d\mu_1(x) = (1-x^2)^{\alpha}dx$, where $\alpha > 0$.
4. $d\mu_0(x) = (\xi^2-x^2)(1-x^2)^{\alpha-1}dx$ and $d\mu_1(x) = (1-x^2)^{\alpha}dx$, where $|\xi| \geq 1$ and $\alpha > 0$.
5. $d\mu_0(x) = dx + M\delta_1 + M\delta_{-1}$ and $d\mu_1(x) = dx$, where $M \geq 0$.

Generalized coherent pairs

We can restate (12) as

$$S_n(x;\lambda) + b_{n-1}(\lambda)S_{n-1}(x;\lambda) = P_n(x;d\mu_0) + \widehat{a}_{n-1}P_{n-1}(x;d\mu_0), \quad n \geq 1. \quad (12')$$

Let $\mathsf{S}_n(x)$ denote the left hand side of (12'). Clearly S_n' can be expanded in terms of $\{P_k(\cdot;d\mu_1)\}$,

$$\mathsf{S}_n'(x) = nP_{n-1}(x;d\mu_1) + \sum_{k=0}^{n-2} d_{k,n}P_k(x;d\mu_1), \qquad d_{k,n} = \frac{\langle \mathsf{S}_n', P_k(\cdot;d\mu_1)\rangle_{d\mu_1}}{\|P_k(\cdot;d\mu_1)\|_{d\mu_1}^2}.$$

We can conclude the following relation between $\{P_n(\cdot;d\mu_0)\}$ and $\{P_n(\cdot;d\mu_1)\}$

$$P_n(x;d\mu_1) + b_{n-1}P_{n-1}(x;d\mu_1) = \frac{P_{n+1}'(x;d\mu_0)}{n+1} + a_n\frac{P_n'(x;d\mu_0)}{n}, \quad n \geq 1. \quad (13)$$

Generalized coherent pairs

Definition

The pair $\{d\mu_0, d\mu_1\}$ is called a generalized coherent pair if (13) holds for all $n \geq 1$, and this definition extends to the linear functionals $\{\mathcal{U}_0, \mathcal{U}_1\}$.

Theorem

If $\{\mathcal{U}_0, \mathcal{U}_1\}$ is a generalized coherent pair, then at least one of them has to be semiclassical of class at most 1 and the other one is a rational perturbation of it. (Delgado, Marcellán, 2004)

All generalized coherent pairs of linear functionals are listed in such a paper.

Some examples of symmetrically generalized coherent pairs have been studied by E. X. L. Andrade , C. F. Bracciali and A. Sri Ranga (2009) when $\mathcal{U}_0$ is associated with the Gegenbauer weight and $\mathcal{U}_1 = \frac{1-x^2}{1+qx^2}\mathcal{U}_0$ + two Dirac deltas with the same masses and supported at the zeros of $1 + qx^2$ if $-1 \leq q < 0$. If $q > 0$ you have not mass points.

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Sobolev type OP

An inner product is called a Sobolev type inner product if the derivatives appear only on function evaluations on a finite discrete set. More precisely, such an inner product takes the form

$$\langle f, g \rangle_S := \int_{\mathbb{R}} f(x)g(x)d\mu_0 + \sum_{k=1}^{m} \int_{\mathbb{R}} f^{(k)}(x)g^{(k)}(x)d\mu_k, \tag{14}$$

where $d\mu_0$ is a positive Boreal measure supported on an infinite subset of the real line and $d\mu_k$, $k = 1, 2, ..., m$, are positive Borel measures supported on finite subsets of the real line.

Sobolev type OP

The first study was carried out for the classical weight functions. The Laguerre case was studied by Koekoek (1990) with $d\mu_0 = x^\alpha e^{-x}dx$, $\alpha > -1$, and $d\mu_k = M_k\delta_0, k = 1, 2, ..., m$; the n-th Sobolev orthogonal polynomial, S_n, is given by

$$S_n(x) = \sum_{k=0}^{\min\{n,m+1\}} (-1)^k A_{n,k} L_{n-k}^{(\alpha+k)}(x),$$

in which A_k are constants determined by a linear system of equations. The Gegenbauer case was studied by Bavinck and Meijer (1989/90) with $d\mu_0 = (1-x^2)^{\lambda-1/2}dx + A(\delta_{-1}+\delta_1)$, $\lambda > -1/2$, and $m = 1$, $d\mu_1 = B(\delta_{-1}+\delta_1)$; the n-th Sobolev orthogonal polynomial is given by

$$S_n(x) = \sum_{k=0}^{2} a_{k,n} x^k C_{n-k}^{(\lambda+k)}(x)$$

where $a_{0,n}$, $a_{1,n}$, $a_{2,n}$ are appropriate constants. In both cases, the Sobolev orthogonal polynomials satisfy higher (than three) order recurrence relations that expands $q(x)S_n(x)$ as a sum of S_m.

Sobolev type OP

When $M_k = 0$, $k = 1, 2, ..., m-1$, and $d\mu_m = M_m\delta_c$, the inner product (14) becomes

$$\langle f, g\rangle_m := \int_{\mathbb{R}} f(x)g(x)d\mu_0 + M_m f^{(m)}(c)g^{(m)}(c), \tag{15}$$

where $c \in \mathbb{R}$ and $M_m \geq 0$.

For $i, j \in \mathbb{N}_0$, define

$$K_{n-1}^{(i,j)}(x, y) := \sum_{l=0}^{n-1} \frac{P_l^{(i)}(x)P_l^{(j)}(y)}{||P_l||_{d\mu_0}^2}.$$

Sobolev type OP

It was shown by Marcellán and Ronveaux (1990) that

$$S_n(x) = P_n(x) - \frac{M_m P_n^{(m)}(c)}{1 + M_m K_{n-1}^{(m,m)}(c,c)} K_{n-1}^{(0,m)}(x,c), \tag{16}$$

which extends the expression for $m = 0$ by Krall (1980/81). From this relation, one deduces immediately that

$$S_{n+1}(x) + a_n S_n(x) = P_{n+1}(x) + b_n P_n(x), \qquad n \geq 0,$$

where a_n and b_n are constants that can be easily determined.
The Sobolev polynomials S_n also satisfy a higher order recurrence relation

$$(x-c)^{m+1} S_n(x) = \sum_{j=n-m-1}^{n+m+1} c_{n,j} S_j(x), \tag{17}$$

where $c_{n,n+m+1} = 1$ and $c_{n,n-m-1} \neq 0$.

Sobolev type OP

There are two types of results in this direction, both related to Sobolev orthogonal polynomials.

The first one gives a characterization of an inner product $\langle \cdot, \cdot \rangle$ for which orthogonal polynomials satisfy the recurrence relation of the form (17), which holds if the operator of multiplication by $M_{m,c} := (\cdot - c)^{m+1}$ is symmetric, *i.e.*, $\langle M_{m,c}p, q \rangle = \langle p, M_{m,c}q \rangle$. It was proved by Durán (1993) that if $\langle \cdot, \cdot \rangle$ is an inner product such that $M_{m,c}$ is symmetric and it commutes with the operator $M_{0,c}$, *i.e.*, $\langle M_{m,c}p, M_{0,c}q \rangle = \langle M_{0,c}p, M_{m,c}q \rangle$, then there exists a nontrivial positive Borel measure $d\mu_0$ and a real, positive semi-definite matrix A of size $m+1$, such that the inner product is of the form

$$\langle p, q \rangle = \int_{\mathbb{R}} p(x)q(x)d\mu_0 + \left(p(c), p'(c), \ldots, p^{(m)}(c)\right) A \left(q(c), q'(c), \ldots, q^{(m)}(c)\right)^T .$$

Sobolev type OP

Furthermore, a connection between such Sobolev orthogonal polynomials and matrix orthogonal polynomials was established by Durán and Van Assche (1995), by representing the higher order recurrence relation as a three term recurrence relation with matrix coefficients for a family of matrix orthogonal polynomials defined in terms of the Sobolev orthogonal polynomials.

The second type of Favard type theorem was given by Evans, Littlejohn, Marcellán, Markett and Ronveaux (1995), where it was proved that the operator of multiplication by a polynomial h is symmetric with respect to the inner product (14) if and only if $d\mu_k, k = 1, 2, .., m$, are discrete measures whose supports are related to the zeros of h and their derivatives. Consequently, higher order recurrence relations for Sobolev inner products appear only in Sobolev inner product of the second type.

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Sobolev type inner products

The first work on asymptotics for Sobolev orthogonal polynomials was carried out by Marcellán and Van Assche (1993) for the inner product

$$\langle f, g\rangle_S = \int_{-1}^{1} f(x)g(x)d\mu_0(x) + M_1 f'(c)g'(c),$$

where $c \in \mathbb{R}$, $M_1 > 0$ and the measure $d\mu_0$ belongs to the Nevai class $M(0,1)$. If $c \in \mathbb{R} \setminus \operatorname{supp} \mu_0$, then

$$\lim_{n\to\infty} \frac{S_n(z)}{P_n(z, d\mu_0)} = \frac{(\Phi(z) - \Phi(c))^2}{2\Phi(z)(z-c)}, \qquad \Phi(z) := z + \sqrt{z^2 - 1},$$

locally uniformly outside the support of the measure, where $\sqrt{z^2-1} > 0$ when $z > 1$.

If $c \in \operatorname{supp}\mu_0$, then $\lim_{n\to\infty} \dfrac{S_n(z)}{P_n(z; d\mu_0)} = 1$ outside the support of the measure.

Some Extensions

- Bounded support.

1. $A \in \mathbb{R}^{(N,N)}$

$$\lim_{n\to\infty} \frac{S_n(z)}{P_n(z,d\mu_0)} = \left(\frac{(\Phi(z)-\Phi(c))^2}{2\Phi(z)(z-c)} \right)^r, \qquad r := \mathrm{rankA},$$

locally uniformly outside the support of the measure.

2. López Lagomasino, Marcellán and Van Assche (1995)

$$\langle f, g \rangle = \int f(x)g(x)d\mu_0(x) + \sum_{j=1}^{N} \sum_{k=0}^{N_j} f^{(k)}(c_j) L_{j,k}(g;c_j),$$

where $d\mu_0 \in M(0,1)$, $\{c_k\}_{k=1}^N \in \mathbb{R} \setminus \operatorname{supp} \mu_0$, $j = 1, \ldots, N$, and $L_{j,k}$ is an ordinary differential operator. Then

$$\lim_{n\to\infty} \frac{S_n^{(\nu)}(z)}{P_n^{(\nu)}(z,d\mu_0)} = \prod_{j=1}^{m} \left(\frac{(\Phi(z)-\Phi(c))^2}{2\Phi(z)(z-c)} \right)^{I_j},$$

Some Extensions

- where I_j is the dimension of the square matrix obtained from the matrix of the coefficients of L_{j,N_j} after deleting all zero rows and columns. On the other hand if both the measure $d\mu_0$ and its support Δ are regular, then

$$\limsup_{n\to\infty} \|S_n^{(j)}\|_{\Delta}^{1/n} = C(\Delta), \qquad j \geq 0,$$

where $\|\cdot\|_{\Delta}$ denotes the uniform norm in the support of the measure and $C(\Delta)$ is its logarithmic capacity (López-Lagomasino, Pijeira, 1999).

Some Extensions

- Unbounded support (Marcellán, Moreno-Balcázar, 2006).

$$d\mu_0 = x^{\alpha} e^{-x} dx, \quad \alpha > 1, \quad c = 0, \quad A \in \mathbb{R}^{(2,2)}.$$

1. Outer relative asymptotics.
2. Outer relative asymptotics for scaled polynomials.
3. Mehler-Heine formula.
4. Inner strong asymptotics.

If $c < 0$, then

$$\lim_{n\to\infty} \frac{S_n(z)}{L_n^{(\alpha)}(z)} = \left(\frac{\sqrt{-z} - \sqrt{-c}}{\sqrt{-z} + \sqrt{-c}} \right)^r, \qquad r = \operatorname{rank} A,$$

locally uniformly on compact subsets of the exterior of $\mathbb{R}_+$ (Marcellán, Zejnullahu, Fejzullahu, Huertas, 2012).

Some Extensions

Extensions for $A \in \mathbb{R}^{(N,N)}$, $c = 0$, as well as $c < 0$ and $c > 0$.

1. Outer relative asymptotics.
2. Mehler-Heine formula.

(Alfaro, Moreno-Balcázar, Peña, Rezola, 2011, Marcellán, Pérez-Valero, Quintana, 2014).

Continuous Sobolev inner products

- Bounded support - coherent case (Martínez-Finkelshtein, Moreno-Balcázar, Pérez, Piñar, 1998).

$$\lim_{n\to\infty} \frac{S_n(z)}{P_n(z;d\mu_1)} = \frac{2}{\Phi'(z)},$$

  locally uniformly outside $[-1,1]$.

- If the measures μ_0 and μ_1 are absolutely continuous and belong to the Szegő class, the above result is also true (Martínez-Finkelshtein, 2000). The role is played by the measure $d\mu_1$!!

- Unbounded support - coherent case.
  1. Outer relative asymptotics.
  2. Scaled outer asymptotics.
  3. Inner strong asymptotics.

- The Freud Sobolev case (Geronimo, Lubinsky, Marcellán, 2005).

Contents

Sobolev OP of several variables

For $x \in \mathbb{R}^d$ and $\alpha \in \mathbb{N}_0^d$, the (total) degree of the monomial x^α is, by definition, $|\alpha| := \alpha_1 + \cdots + \alpha_d$. Let Π_n^d denote the space of polynomials of total degree at most n in d-variables. It is known that $\dim \Pi_n^d = \binom{n+d}{n}$.

Let Π^d denote the space of all polynomials in d-variables. Let $\langle \cdot, \cdot \rangle$ be an inner product defined on $\Pi^d \times \Pi^d$. A polynomial $P \in \Pi_n^d$ is orthogonal if $\langle P, q \rangle = 0$ for all $q \in \Pi_{n-1}^d$. For $n \in \mathbb{N}_0^d$, let $\mathcal{V}_n^d$ denote the space of polynomials of total degree n. Then $\dim \mathcal{V}_n^d = \binom{n+d-1}{n}$. In contrast to one-variable, the space $\mathcal{V}_n^d$ can have many different bases when $d \geq 2$. Moreover, the elements in $\mathcal{V}_n^d$ may not be orthogonal to each other.

For the structure and properties of orthogonal polynomials in several variables, we refer to Dunkl and Xu (2014).

Sobolev OP of several variables

Example For $\mu > -1$, let $\varpi_\mu(x) = (1 - \|x\|^2)^{\mu - 1/2}$ be the weight function defined on the unit ball $\mathbb{B}^d = \{x \in \mathbb{R}^d : \|x\| \leq 1\}$, where $\|x\|$ denotes the Euclidean norm of $x \in \mathbb{R}^d$. Orthogonal polynomials with respect to ϖ_μ can be given in several different formulations. We give one basis of $\mathcal{V}_n^d(\varpi_\mu)$ in terms of the Jacobi polynomials and spherical harmonics in the spherical coordinates $x = r\xi$, where $0 < r \leq 1$ and $\xi \in \mathbb{S}^{d-1} = \{x : \|x\| = 1\}$. For $0 \leq j \leq n/2$ and $1 \leq j \leq a_{n-2j}^d$, define

$$P_{j,\nu}^n(x) := P_j^{(\mu, n-2j+\frac{d-2}{2})}(2\,\|x\|^2 - 1)\, Y_\nu^{n-2j}(x), \tag{18}$$

where $\{Y_\nu^{n-2j} : 1 \leq \nu \leq a_{n-2j}^d\}$ is an orthonormal basis of $\mathcal{H}_{n-2j}^d$. Then the set $\{P_{j,\ell}^{\mu,n}(x) : 0 \leq j \leq \frac{n}{2},\ 1 \leq \ell \leq a_{n-2j}^d\}$ is a mutually orthogonal basis of $\mathcal{V}_n^d(\varpi_\mu)$. The elements of $\mathcal{V}_n^d$ are eigenfunctions of a second order differential operator $\mathcal{D}_\mu$. More precisely, we have

$$\mathcal{D}_\mu P = -(n+d)(n+2\mu)P, \qquad \forall P \in \mathcal{V}_n^d(\varpi_\mu), \tag{19}$$

where

$$\mathcal{D}_\mu := \Delta - \sum_{j=1}^d \frac{\partial}{\partial x_j} x_j \left[2\mu + \sum_{i=1}^d x_i \frac{\partial}{\partial x_i}\right].$$

Sobolev OP on the unit ball

The first work in the unit ball by Xu (2006) deals with the inner product

$$\langle f, g \rangle_\Delta := \int_{\mathbb{B}^d} \Delta \left[(1 - \|x\|^2) f(x)\right] \Delta \left[(1 - \|x\|^2) g(x)\right] dx,$$

which arises from numerical solution of the Poisson equation studied by Atkinson and Hansen (2005). The geometry of the ball and (18) suggests that one can look for a mutually orthogonal basis of the form

$$q_j(2\|x\|^2 - 1) Y_\nu^{n-2j}(x), \qquad Y_\nu^{n-2j} \in \mathcal{H}_{n-2j}^d, \tag{20}$$

where q_j is a polynomial of degree j in one variable. Such a basis was constructed by Xu (2006) for the space $\mathcal{V}_n^d(\Delta)$ with respect to $\langle \cdot, \cdot \rangle_\Delta$. As a result, it was shown that

$$\mathcal{V}_n^d(\Delta) = \mathcal{H}_n^d \oplus (1 - \|x\|^2) \mathcal{V}_{n-2}(\varpi_2).$$

Sobolev OP on the unit ball

The next inner product considered on the ball, which should be, in retrospect, the first one being considered, is defined by

$$\langle f, g \rangle_{-1} := \lambda \int_{\mathbb{B}^d} \nabla f(x) \cdot \nabla g(x) dx + \int_{\mathbb{S}^{d-1}} f(\xi) g(\xi) d\sigma(\xi),$$

where $\lambda > 0$. An alternative is to replace the integral over $\mathbb{S}^{d-1}$ by $f(0)g(0)$. A basis of the form (20) was constructed explicitly by Xu (2008) for the space $\mathcal{V}_n^d(\Delta)$ with respect to $\langle \cdot, \cdot \rangle_{-1}$, from which it follows that

$$\mathcal{V}_n^d(\varpi_{-1}) = \mathcal{H}_n^d \oplus (1 - \|x\|^2) \mathcal{V}_{n-2}(\varpi_1). \tag{21}$$

The main part of the basis, those in $(1 - \|x\|^2)\mathcal{V}_{n-2}(\varpi_1)$, can be given in terms of the Jacobi polynomials $P_n^{(-1,b)}$ of negative index, which explains why we used the notation ϖ_{-1}. Another interesting aspect of this case is that the polynomials in $\mathcal{V}_n^d(\varpi_{-1})$ are eigenfunctions of the differential operator $\mathcal{D}_{-1}$.

Sobolev OP on the unit ball

For $k \in \mathbb{N}$, the equation $\mathcal{D}_{-k} Y = \lambda_n Y$ of (19) was studied by Piñar and Xu (2009), where a complete system of polynomial solutions was determined explicitly. For $k \geq 2$, however, it is not known if the solutions are Sobolev orthogonal polynomials. Closely related to the case of $k = 2$ is the following inner product

$$\langle f, g \rangle_{-2} := \lambda \int_{B^d} \Delta f(x) \Delta g(x) dx + \int_{S^{d-1}} f(x) g(x) d\sigma, \quad \lambda > 0.$$

An explicit basis for the space $\mathcal{V}_n^d(\varpi_{-2})$ of the Sobolev orthogonal polynomials with respect to $\langle \cdot, \cdot \rangle_{-2}$ was constructed by Piñar and Xu (2009), from which it follows that

$$\mathcal{V}_n^d(\varpi_{-2}) = \mathcal{H}_n^d \oplus (1 - \|x\|^2) \mathcal{H}_{n-2}^d \oplus (1 - \|x\|^2)^2 \mathcal{V}_{n-4}^d(\varpi_2). \tag{22}$$

The main part of the base, those in $(1 - \|x\|^2)^2 \mathcal{V}_{n-2}(\varpi_2)$, can be given in terms of the Jacobi polynomials $P_n^{(-2,b)}$ of negative index.

Sobolev OP on the unit ball

It turns out that the Sobolev orthogonal polynomials for the last two cases can be used in the study of the spectral methods for numerical solution of partial differential equations. This connection was established by Li and Xu (2013), where, for $s \in \mathbb{N}$, the following inner product in the Sobolev space $W_p^s(\mathbb{B}^d)$ is defined,

$$\langle f, g \rangle_{-s} := \langle \nabla^s f, \nabla^s g \rangle_{\mathbb{B}^d} + \sum_{k=0}^{\lceil \frac{\varrho}{2} \rceil - 1} \lambda_k \langle \Delta^k f, \Delta^k g \rangle_{\mathbb{S}^{d-1}}, \tag{23}$$

where λ_k, $k = 0, 1 \ldots, \lceil \frac{\varrho}{2} \rceil - 1$, are positive constants, and

$$\nabla^{2m} := \Delta^m \quad \text{and} \quad \nabla^{2m+1} := \nabla \Delta^m, \qquad m = 1, 2, \ldots.$$

For $s > 2$, the space $\mathcal{V}_n^d(\varpi_{-s})$ associated with $\langle \cdot, \cdot \rangle_{-s}$ cannot be decomposed as in (21) and (22). Nevertheless, an explicit mutually orthogonal basis was constructed by Li and Xu (2014), which requires considerable effort, and the basis uses extension of the Jacobi polynomials $P_n^{(\alpha,\beta)}$ for $\alpha, \beta \in \mathbb{R}$ that avoids the degree reduction when $-\alpha - \beta - n \in \{0, 1, \ldots, n\}$.

Sobolev OP on the unit ball

The main result by Li and Xu (2014) establishes an estimate for the polynomial approximation in the Sobolev space $\mathcal{W}_p^s(\mathbb{B}^d)$, the proof relies on the Fourier expansion in the Sobolev orthogonal polynomials associated with (23).
Another Sobolev inner product considered on the unit ball is defined by

$$\langle f, g \rangle = \int_{\mathbb{B}^d} \nabla f(x) \cdot \nabla g(x) W_\mu(x) dx + \lambda \int_{\mathbb{B}^d} f(x) g(x) W_\mu(x) dx,$$

which is an extension of the Sobolev inner product of the coherent pair in the case of the Gegenbauer weight of one-variable. A mutually orthogonal basis was constructed Pérez, Piñar and Xu (2013), which has the form of (20) but the corresponding q_j is orthogonal with respect to a rather involved Sobolev inner product of one variable.

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Thanks for your attention