Properties of certain classes of semi-classical orthogonal polynomials

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Outline

- Semi-classical orthogonal polynomials
  - Semiclassical Laguerre polynomials
  - Generalized Freud polynomials
- Structural relations satisfied by orthogonal polynomials
  - A problem due to Askey
  - A conjecture due to Ismail
  - An application

The pioneer of orthogonality

The pioneer of orthogonality

Murphy [1835] first defined orthogonal functions but it was Tchebychev who realised their importance.

His work since 1855 was motivated by the analogy with Fourier Series and by the theory of continued fractions and approximation theory.

The Tchebychev polynomials

$$T_n(x) = \cos n\theta \quad \text{where} \quad x = \cos\theta \quad \text{for} \quad n \in \mathbb{N}.$$

$$\int_0^\pi \cos m\theta \ \cos n\theta \ d\theta = \begin{cases} 0, & n \neq m \\ \frac{\pi}{2}, & m = n. \end{cases}$$

Making the substitution $x = \cos\theta$ in this integral, then $dx = -\sin\theta \ d\theta$ or

$$d\theta = \frac{-dx}{\sin\theta} = \frac{-dx}{\sqrt{1-x^2}}.$$

Also when $\theta = 0$, $x = 1$ and $\theta = \pi$, $x = -1$ so

$$\int_0^\pi \cos m\theta \ \cos n\theta \ d\theta = \int_{-1}^1 T_n(x) T_m(x) (1 - x^2)^{-1/2} dx$$
$$= \begin{cases} 0, & n \neq m \\ \frac{\pi}{2}, & m = n. \end{cases}$$

Definition

A sequence of real polynomials $\{P_n\}_{n=0}^{\infty}$, $\deg(P_n) = n$, is orthogonal with respect to a positive measure μ with support S defined on $\mathbb{R}$, if

$$\int_S P_m(x)P_n(x)d\mu(x) = \begin{cases} 0, & n \neq m \\ h_n \neq 0, & n = m, \end{cases}$$

provided that moments of all order exist.

For Tchebychev polynomials

$$\int_{-1}^{1} T_n(x)T_m(x)(1-x^2)^{-1/2}dx = \begin{cases} 0, & n \neq m \\ \frac{\pi}{2}, & n = m. \end{cases}$$

The measure $\mu(x)$ is absolutely continuous and can be expressed ito a non-negative weight $w(x)$, with support on $[-1, 1] \in \mathbb{R}$, $d\mu(x) = w(x)dx$.

Tchebychev polynomials $\{T_n(x)\}_{n=0}^{\infty}$ are orthogonal on the interval $[-1, 1]$ with respect to the positive weight function $w(x) = (1-x^2)^{-1/2}$.

Three-term recurrence for monic polynomials

A sequence $\{p_n\}$ of monic polynomials orthogonal with respect to a positive measure μ satisfies a three-term recurrence relation

$$p_{n+1} = (x - \alpha_n)p_n - \beta_n p_{n-1}, \quad n = 0, 1, 2, \ldots$$

with initial conditions

$$p_{-1} \equiv 0, \qquad p_0 \equiv 1$$

and recurrence coefficients

$$\alpha_n \in \mathbb{R}, \; n = 0, 1, 2 \ldots, \quad \beta_n > 0, \quad n = 1, 2, \ldots.$$

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The coefficients in the three-term recurrence relation can also be expressed in terms of determinants whose entries are the moments

$$\mu_k = \int_a^b x^k d\mu$$

associated with measure μ.

Hankel determinants

$$\alpha_n = \frac{\widetilde{\Delta}_{n+1}}{\Delta_{n+1}} - \frac{\widetilde{\Delta}_n}{\Delta_n}, \qquad \beta_n = \frac{\Delta_{n+1}\Delta_{n-1}}{\Delta_n^2},$$

where Δ_n is the Hankel determinant

$$\Delta_n = \det\left[\mu_{j+k}\right]_{j,k=0}^{n-1} = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-1} \\ \mu_1 & \mu_2 & \cdots & \mu_n \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-2} \end{vmatrix}, \qquad n \geq 1,$$

with $\Delta_0 = 1$, $\Delta_{-1} = 0$, and $\widetilde{\Delta}_n$ is the determinant

$$\widetilde{\Delta}_n = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-2} & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n-1} & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-3} & \mu_{2n-1} \end{vmatrix}, \qquad n \geq 1,$$

with $\widetilde{\Delta}_0 = 0$ and μ_k is the kth moment.

The converse

Theorem (Spectral Theorem for orthogonal polynomials)

Consider a family of monic polynomials satisfies a three-term recurrence relation

$$p_{n+1} = (x - \alpha_n)p_n - \beta_n p_{n-1}$$

with initial conditions $p_0 = 1$ *and* $p_{-1} = 0$ *where* $\alpha_{n-1} \in \mathbb{R}$ *and* $\beta_n > 0$, $n \in \mathbb{N}$.

Then there exists a measure μ *on the real line such that these polynomials are monic orthogonal polynomials satisfying*

$$\int_{\mathbb{R}} p_n(x)p_m(x)\,d\mu(x) = \begin{cases} 0, & n \neq m \\ h_n \neq 0, & n = m, \end{cases} \qquad m, n = 0, 1, 2, \ldots$$

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- Proof does not give explicit information about measure or support.
- Measure need not be unique: depends on Hamburger moment problem.

The inverse problem: Given a three-term recurrence relation

Operator theory plays an important role in answering questions concerning

- uniqueness of the orthogonality measure
- nature of the orthogonality measure
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Extracting information from the measure regarding characterising properties of the orthogonal polynomials, such as the coefficients of the

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For most classical orthogonal polynomials, the properties satisfied are known.

Semi-classical Laguerre weight

Consider monic orthogonal polynomials with respect to the semi-classical Laguerre weight

$$w(x;t) = x^{\lambda} \exp(-x^2 + tx), \quad x \in \mathbb{R}^+, \; \lambda > -1, t \in \mathbb{R} \tag{1}$$

which satisfy the three-term recurrence relation

$$xL_n(x;t) = L_{n+1}(x;t) + \widetilde{\alpha}_n(t)L_n(x;t) + \widetilde{\beta}_n(t)L_{n-1}(x;t), \tag{2}$$

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Theorem (Boelen and van Assche, 2011)

The coefficients $\widetilde{\alpha}_n(t)$ and $\widetilde{\beta}_n(t)$ in the recurrence relation (2) associated with the semi-classical Laguerre weight (1) satisfy the discrete system

$$(2\widetilde{\alpha}_n - t)(2\widetilde{\alpha}_{n-1} - t) = \frac{(2\widetilde{\beta}_n - n)(2\widetilde{\beta}_n - n - \lambda)}{\widetilde{\beta}_n},$$

$$2\widetilde{\beta}_n + 2\widetilde{\beta}_{n+1} + \widetilde{\alpha}_n(2\widetilde{\alpha}_n - t) = 2n + \lambda + 1.$$

Theorem (Fillipuk, van Assche & Zhang, 2012)

The coefficients $\widetilde{\alpha}_n(t)$ in the recurrence relation (2) associated with the semi-classical Laguerre weight (1) are given by

$$\widetilde{\alpha}_n(t) = \tfrac{1}{2} q_n(z) + \tfrac{1}{2} t,$$

with $z = \frac{1}{2}t$ where $q_n(z)$ satisfies

$$\frac{d^2 q_n}{dz^2} = \frac{1}{2q_n}\left(\frac{d\,q_n}{dz}\right)^2 + \tfrac{3}{2} q_n^3 + 4z q_n^2 + 2(z^2 - 2n - \lambda - 1)q_n - \frac{2\lambda^2}{q_n},$$

which is $\mathrm{P_{IV}}$, with parameters $(A, B) = (2n + \lambda + 1, -2\lambda^2)$.

Remark

The parameters (A, B) satisfy the condition for $\mathrm{P_{IV}}$ to have solutions expressible in terms of parabolic cylinder functions

Semi-classical Laguerre weight

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We first derive an explicit expression for the moment $\mu_0(t;\lambda)$.

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Theorem (Clarkson & J, 2014)

For the weight (1), the moment $\mu_0(t;\lambda)$ is given by

$$\mu_0(t;\lambda) = \begin{cases} \dfrac{\Gamma(\lambda+1)\exp\left(\frac{1}{8}t^2\right)}{2^{(\lambda+1)/2}} D_{-\lambda-1}\left(-\frac{1}{2}\sqrt{2}\,t\right), & \text{if} \quad \lambda \notin \mathbb{N}, \\ \frac{1}{2}\sqrt{\pi}\,\dfrac{d^m}{dt^m}\left\{\exp\left(\frac{1}{4}t^2\right)\left[1+\operatorname{erf}\left(\frac{1}{2}t\right)\right]\right\}, & \text{if} \quad \lambda = m \in \mathbb{N}, \end{cases}$$

with $D_\nu(\zeta)$ the parabolic cylinder function and $\operatorname{erf}(z)$ the error function. Further $\mu_0(t;\lambda)$ satisfies the equation

$$\frac{d^2\mu_0}{dt^2} - \frac{1}{2}t\frac{d\mu_0}{dt} - \frac{1}{2}(\lambda+1)\mu_0 = 0.$$

The weight has the form

$$w(x;t) = w_0(x)\exp(xt), \qquad x \in [a,b], \tag{3}$$

and

$$\mu_k = \int_a^b x^k w_0(x)\exp(xt)\,dx = \frac{d^k}{dt^k}\left(\int_a^b w_0(x)\exp(xt)\,dx\right) = \frac{d^k \mu_0}{dt^k}.$$

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Theorem (Clarkson & J, 2014)

If the weight has the form (3), then the Hankel determinant is given by $\Delta_n(t) = \mathcal{W}\left(\mu_0, \frac{d\mu_0}{dt}, \ldots, \frac{d^{n-1}}{dt^{n-1}}\mu_0\right)$, $\Delta_0 = 1$, $\Delta_{-1} = 0$ *where* $\mathcal{W}(\varphi_1, \varphi_2, \ldots, \varphi_n)$ *is the Wronskian given by*

$$\mathcal{W}(\varphi_1, \varphi_2, \ldots, \varphi_n) = \begin{vmatrix} \varphi_1 & \varphi_2 & \cdots & \varphi_n \\ \varphi_1^{(1)} & \varphi_2^{(1)} & \cdots & \varphi_n^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_1^{(n-1)} & \varphi_2^{(n-1)} & \cdots & \varphi_n^{(n-1)} \end{vmatrix}, \qquad \varphi_j^{(k)} = \frac{d^k\varphi_j}{dt^k}.$$

Theorem (Clarkson & J, 2014)

The recurrence coefficients $\widetilde{\alpha}_n(t)$ and $\widetilde{\beta}_n(t)$ associated with the weight (1) are

$$\widetilde{\alpha}_n(t) = \tfrac{1}{2} q_n(z) + \tfrac{1}{2} t,$$

$$\widetilde{\beta}_n(t) = -\tfrac{1}{8} \frac{d\, q_n}{dz} - \tfrac{1}{8} q_n^2(z) - \tfrac{1}{4} z q_n(z) + \tfrac{1}{4}\lambda + \tfrac{1}{2} n,$$

with $z = \tfrac{1}{2} t$, where

$$q_n(z) = -2z + \frac{d}{dz} \ln \frac{\Psi_{n+1,\lambda}(z)}{\Psi_{n,\lambda}(z)}$$

$$\Psi_{n,\lambda}(z) = \mathcal{W}\left(\psi_\lambda, \frac{d\psi_\lambda}{dz}, \ldots, \frac{d^{n-1}\psi_\lambda}{dz^{n-1}}\right), \quad \Psi_{0,\lambda}(z) = 1,$$

and

$$\psi_\lambda(z) = \begin{cases} D_{-\lambda-1}\big(-\sqrt{2}\, z\big) \exp\left(\tfrac{1}{2} z^2\right), & \text{if} \quad \lambda \notin \mathbb{N}, \\ \dfrac{d^m}{dz^m} \left\{ \left[1 + \operatorname{erf}(z)\right] \exp(z^2) \right\}, & \text{if} \quad \lambda = m \in \mathbb{N}. \end{cases}$$

Theorem (Clarkson & J, 2016)

Let $L_n(x;t)$ denote the monic semi-classical Laguerre polynomials orthogonal with respect to

$$w(x;t) = x^{\lambda}\exp(-x^2+tx), \quad x\in\mathbb{R}^+.$$

Then, for $\lambda > -1$ and $t\in\mathbb{R}$, the zeros $x_{1,n} < x_{2,n} < \cdots < x_{n,n}$ of $L_n(x;t)$

(i) *are real, distinct and interlacing;*

(ii) *strictly increase with both t and λ;*

(iii) *satisfy*

$$a_n < x_{1,n} < \widetilde{\alpha}_{n-1} < x_{n,n} < b_n,$$

where

$$a_n = \min_{1\le k\le n-1}\left\{\tfrac{1}{2}(\widetilde{\alpha}_k+\widetilde{\alpha}_{k-1}) - \tfrac{1}{2}\sqrt{(\widetilde{\alpha}_k+\widetilde{\alpha}_{k-1})^2+4c_n\widetilde{\beta}_k}\right\},$$

$$b_n = \max_{1\le k\le n-1}\left\{\tfrac{1}{2}(\widetilde{\alpha}_k+\widetilde{\alpha}_{k-1}) + \tfrac{1}{2}\sqrt{(\widetilde{\alpha}_k+\widetilde{\alpha}_{k-1})^2+4c_n\widetilde{\beta}_k}\right\},$$

with $c_n = 4\cos^2\left(\frac{\pi}{n+1}\right)+\varepsilon$, $\varepsilon > 0$.

Proof

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we have

$$\frac{\partial}{\partial \lambda} \ln w(x;t) = \ln x,$$

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Similarly, since

$$\frac{\partial}{\partial t} \ln w(x;t) = x,$$

increases with x, it follows that the zeros of $L_n^{(\lambda)}(x;t)$ increase as t increases.

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Theorem

Let $\{p_n\}_{n=0}^{\infty}$ be a sequence of polynomials orthogonal on the interval (c, d). Fix $k, n \in$ with $k < n-1$ and suppose deg(g_{n-k-1})$= n-k-1$ with

$$\pi(x)g_{n-k-1}(x) = G_k(x)p_{n-1}(x) + H(x)p_n(x)$$

where $\pi(x) \neq 0$ for $x \in (c, d)$ and deg$(G_k) = k$.

Then the largest (smallest) zero of G_k is a strict lower (upper) bound for the largest (smallest) zero of p_n.

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Then the largest (smallest) zero of G_k is a strict lower (upper) bound for the largest (smallest) zero of p_n.

Consider the three term recurrence relation

$$\widetilde{\beta}_{n-1}(t)L_{n-2}^{(\lambda)}(x;t) = [x - \widetilde{\alpha}_{n-1}(t)]L_{n-1}^{(\lambda)}(x;t) - L_n^{(\lambda)}(x;t)$$

noting that $\widetilde{\beta}_{n-1}(t)$ does not depend on x.

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These outer bounds a_n and b_n for the extreme zeros follow from

- an approach based on a theorem due to Wall and Wetzel [Ismail and Li, 1992]
- using finite chain sequences
- applying their results to the three term recurrence relation.

Generalized Freud weight

- Polynomials orthogonal with respect to a symmetric moment functional can be generated via quadratic transformation from the classical orthogonal polynomials.

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- Symmetrizing the semi-classical Laguerre weight

$$w(x;t) = x^{\lambda}\exp(-x^2+tx), \quad x\in(0,\infty) \;\text{ for } \lambda>-1 \text{ and } t\in\mathbb{R}$$

yields a sequence $\{S_n(x;t)\}_{n=0}^{\infty}$ of polynomials orthogonal with respect to the even weight

$$w(x;t) = |x|^{2\lambda+1}\exp(-x^4+tx^2) \quad x\in\mathbb{R} \;\text{ for } \lambda>-1 \text{ and } t\in\mathbb{R}$$

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- Semi-classical orthogonal polynomials w.r.t. the generalized Freud weight

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$$S_{n+1}(x;t) = xS_n(x;t) - \beta_n(t;\lambda)S_{n-1}(x;t), \quad S_{-1}=0, \quad S_0=1$$

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The first moment, $\mu_0(t;\lambda)$, can be obtained using the integral representation of a parabolic cylinder function.

$$\begin{aligned}\mu_0(t;\lambda) &= \int_{-\infty}^{\infty} |x|^{2\lambda+1} \exp\left(-x^4 + tx^2\right) dx \\ &= \frac{\Gamma(\lambda+1)}{2^{(\lambda+1)/2}} \exp\left(\tfrac{1}{8}t^2\right) D_{-\lambda-1}\left(-\tfrac{1}{2}\sqrt{2}\,t\right).\end{aligned}$$

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The even moments are

$$\begin{aligned}\mu_{2n}(t;\lambda) &= \int_{-\infty}^{\infty} x^{2n} |x|^{2\lambda+1} \exp\left(-x^4 + tx^2\right) dx \\ &= \frac{d^n}{dt^n}\mu_0(t,\lambda), \quad n = 1, 2, \ldots\end{aligned}$$

and, since the integrand is odd, the odd ones are

$$\mu_{2n+1}(t;\lambda) = 0, \quad n = 1, 2, \ldots$$

Recurrence coefficients of generalized Freud polynomials

The explicit expressions for the recurrence coefficients

$$\beta_n = \frac{\Delta_{n+1}\Delta_{n-1}}{\Delta_n^2}$$

in the three term recurrence relation for generalized Freud polynomials are given by

$$\beta_{2n} = \frac{d}{dt} \ln \frac{\tau_n(t;\lambda+1)}{\tau_n(t;\lambda)},$$
$$\beta_{2n+1} = \frac{d}{dt} \ln \frac{\tau_{n+1}(t;\lambda)}{\tau_n(t;\lambda+1)},$$

for $n \geq 0$, where $\tau_n(t;\lambda)$ is the Hankel determinant given by

$$\tau_n(t;\lambda) = \det \left[\frac{d^{j+k}}{dt^{j+k}} \mu_0(t;\lambda) \right]_{j,k=0}^{n-1}, \quad \tau_0(t;\lambda) = 1.$$

Theorem (Clarkson, J & Kelil, 2016)

For $\lambda > 0$, the recurrence coefficients $\beta_n(t;\lambda)$ also satisfy the nonlinear difference equation

$$\beta_n\left(\beta_{n+1} + \beta_n + \beta_{n-1} - \tfrac{1}{2}t\right) = \frac{2n + (2\lambda+1)[1-(-1)^n]}{8}, \qquad (4)$$

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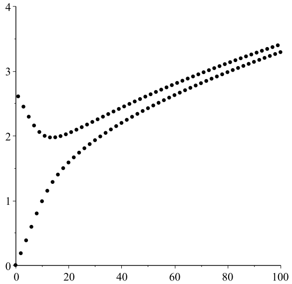
An overview of the problem of existence and uniqueness of positive solutions of nonlinear difference equations of type (4) is given by Alsulami, Nevai, Szabados and van Assche [2015].

Theorem (Clarkson & J, 2018)

For $t \in \mathbb{R}$ and $\beta_0 = 0$, there exists a unique $\beta_1(t;\lambda) > 0$ such that $\{\beta_n(t;\lambda)\}_{n\in\mathbb{N}}$ defined by the nonlinear difference equation (4) is a positive sequence and the solution arises when

$$\beta_1(t;\lambda) = \tfrac{1}{2}t + \tfrac{1}{2}\sqrt{2}\,\frac{D_{-\lambda}\left(-\tfrac{1}{2}\sqrt{2}\,t\right)}{D_{-\lambda-1}\left(-\tfrac{1}{2}\sqrt{2}\,t\right)} = \Phi_\lambda(t).$$

The solution of the nonlinear discrete equation (4) is highly sensitive to the initial conditions $\beta_0(t;\lambda) = 0$ and $\beta_1(t;\lambda) = \Phi_\lambda(t)$.



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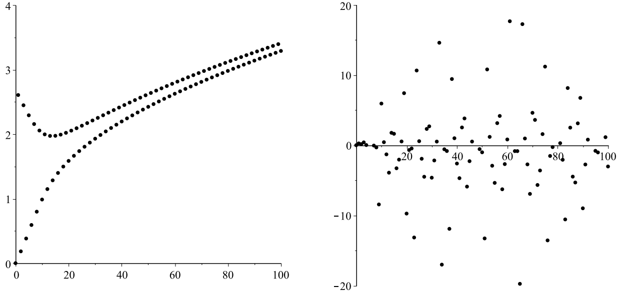

$\beta_1 = \Phi_\lambda(5)$ $\quad$ $\beta_1 = \Phi_\lambda(5) + 10^{-4}$

Figure: Plots of the points (n, β_n) where β_n satisfies (4) with initial conditions $\beta_0 = 0$ and $\beta_1 = \Phi_\lambda(t) + \epsilon$, with $\epsilon = 0, 10^{-4}$, for $t = 5$ and $\lambda = \frac{1}{2}$.

Asymptotic properties for $\beta_n(t;\lambda)$

The asymptotic expansion of $\beta_n(t;\lambda)$ satisfying the nonlinear discrete equation

$$\beta_n\left(\beta_{n+1}+\beta_n+\beta_{n-1}-\tfrac{1}{2}t\right)=\frac{2n+(2\lambda+1)[1-(-1)^n]}{8},$$

when

- $t=0$ and $\lambda=-\frac{1}{2}$ was studied by Lew and Quarles [1983];
- $t\in\mathbb{R}$ and $\lambda=-\frac{1}{2}$ was given by Clarke and Shizgal [1993];

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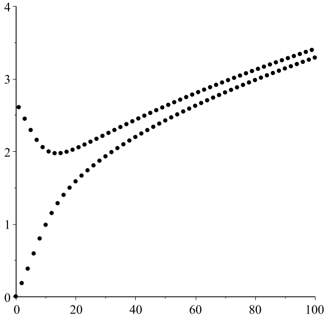
Theorem (Clarkson & J, 2018)

Let $t,\lambda\in\mathbb{R}$, then as $n\to\infty$, the recurrence coefficient β_n associated with monic generalized Freud polynomials

$$\beta_{2n}(t;\lambda)=\frac{\sqrt{6}\,n^{1/2}}{6}\left\{1+\frac{\sqrt{6}\,t}{12n^{1/2}}+\frac{t^2}{48n}-\frac{t^4-48}{4608n^2}+\mathcal{O}(n^{-5/2})\right\},$$

$$\begin{aligned}\beta_{2n+1}(t;\lambda)=\frac{\sqrt{3}\,(2n+1)^{1/2}}{6}&\left\{1+\frac{\sqrt{3}\,t}{6(2n+1)^{1/2}}+\frac{t^2+12(2\lambda+1)}{24(2n+1)}\right.\\&\left.-\frac{t^4+24(2\lambda+1)t^2+96(6\lambda^2+6\lambda+1)}{1152(2n+1)^2}+\mathcal{O}(n^{-5/2})\right\}.\end{aligned}$$



4
3
2
1
0
0
20
40
60
80
100

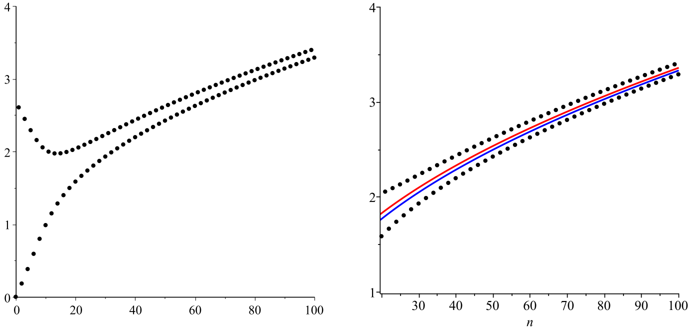


Figure: Plots of points (n, β_n) satisfying (4) with initial conditions $\beta_0 = 0$, $\beta_1 = \Phi_\lambda(t)$ and the first few terms of the asymptotic expansions (5) of $\beta_{2n}(t;\lambda)$ (blue) and $\beta_{2n+1}(t;\lambda)$ (red) with $\lambda = \frac{1}{2}$ and $t = 5$.

As $t \to \infty$, the recurrence coefficient $\beta_n(t;\lambda)$ has the asymptotic expansion

$$\beta_{2n}(t;\lambda) = \frac{n}{t} - \frac{2n(2\lambda - n + 1)}{t^3} + \mathcal{O}(t^{-5}),$$

$$\beta_{2n+1}(t;\lambda) = \frac{t}{2} + \frac{\lambda - n}{t} - \frac{2(\lambda^2 - 4\lambda n + n^2 - \lambda - n)}{t^3} + \mathcal{O}(t^{-5}),$$

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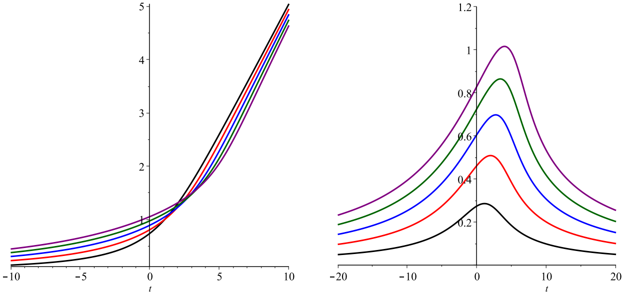
\end{aligned}$$

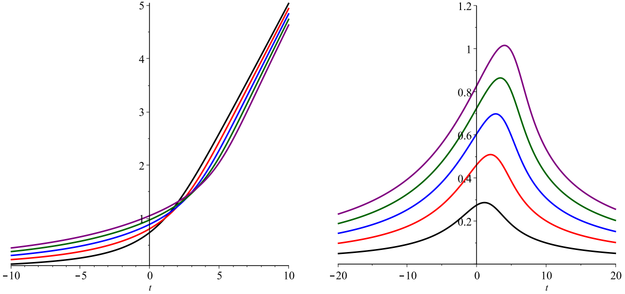
Further, as $t \to -\infty$

$$\begin{aligned}
\beta_{2n}(t;\lambda) &= -\frac{n}{t} + \frac{2n(2\lambda + 3n + 1)}{t^3} + \mathcal{O}(t^{-5}),\\
\beta_{2n+1}(t;\lambda) &= -\frac{\lambda + n + 1}{t} + \frac{2(\lambda + n + 1)(\lambda + 3n + 2)}{t^3} + \mathcal{O}(t^{-5}).
\end{aligned}$$



$\beta_{2n-1}(t;\lambda), \quad n = 1, 2, \dots, 5 \qquad \beta_{2n}(t;\lambda), \quad n = 1, 2, \dots, 5$

Figure: Plots of the recurrence coefficients $\beta_{2n-1}(t;\frac{1}{2})$ and $\beta_{2n}(t;\frac{1}{2})$, for $n = 1$ (black), $n = 2$ (red), $n = 3$ (blue), $n = 4$ (green) and $n = 5$ (purple).



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$$\lim_{t\to\infty} \beta_n(t;\lambda) = \tfrac{1}{4}[1 - (-1)^n]t.$$

Properties of the zeros of generalized Freud polynomials

- Asymptotic properties of the extreme zeros of generalized Freud polynomials were studied by Freud [1986] and Nevai [1986];
- Subsequently, Kasuga and Sakai [2005] extended and generalized these results;
- Arceo, Huertas and Marcellán [2016] generalize the electrostatic interpretation of the zero distribution and provide an equation of motion for the distribution of the zeros of a polynomial associated with an Uvarov modification of a quartic Freud type weight ($\lambda = -\frac{1}{2}$).

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Remark

The generalized Freud weight $w(x;t) = |x|^{2\lambda+1}\exp(-x^4 + tx^2)$, *is even and the zeros of the corresponding orthogonal polynomials are symmetric about the origin.*

Theorem (Clarkson & J, 2016)

Let $S_n(x;t)$ be the monic generalized Freud polynomials orthogonal with respect to the weight $w(x;t) = |x|^{2\lambda+1}\exp(-x^4 + tx^2)$, and let $x_{n,1}(\lambda,t) < x_{n,2}(\lambda,t) < \cdots < x_{n,[n/2]}(\lambda,t)$ denote the positive zeros of $S_n(x;t)$ where $[m]$ is the largest integer smaller than m. Then, for $\lambda > -1$ and $t \in \mathbb{R}$

(i) *the zeros of $S_n(x;t)$ are real and distinct and*

$$x_{n,1}(\lambda,t) < x_{n-1,1}(\lambda,t) < x_{n,2}(\lambda,t) < \cdots < x_{n,[n/2]}(\lambda,t);$$

(ii) *the νth zero $x_{n,\nu}(\lambda,t)$, for a fixed value of ν, is an increasing function of both λ and t;*

(iii) *the largest zero satisfies the inequality*

$$x_{n,[n/2]}(\lambda,t) < \max_{1\le k\le n-1} \sqrt{c_n\beta_k(t;\lambda)},$$

where $c_n = 4\cos^2\left(\frac{\pi}{n+1}\right) + \varepsilon$, $\varepsilon > 0$.

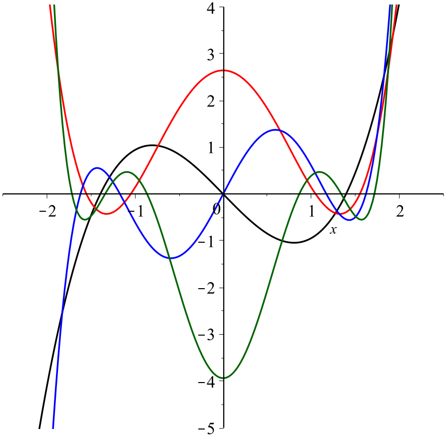

Figure: Plots of the polynomials $S_3(x;t)$ (black), $S_4(x;t)$ (red), $S_5(x;t)$ (blue), $S_6(x;t)$ (green) for $t = 3$, with $\lambda = \frac{1}{2}$.

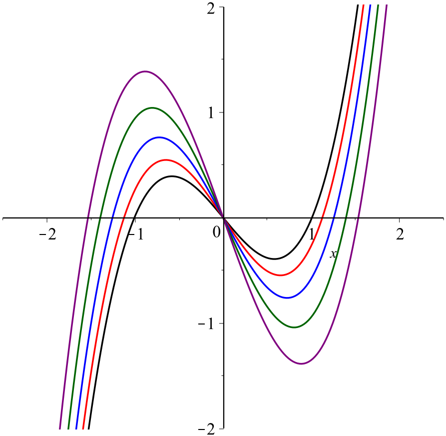

Figure: Plots of the polynomial $S_n(x;t)$, $n = 3$ for $t = 0$ (black), $t = 1$ (red), $t = 2$ (blue), $t = 3$ (green) and $t = 4$ (purple), with $\lambda = \frac{1}{2}$.

The differential-difference relation

Consider the differential-difference equation satisfied by monic orthogonal polynomials $S_n(x; t)$ with respect to the generalized Freud weight

$$\pi(x)\frac{d}{dx}S_n(x;t) = \sum_{j=-t}^{s} a_{n,n+j}S_{n+j}(x;t), \quad n = 1, 2, \ldots \tag{6}$$

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- Shohat [1939] gave a procedure using quasi-orthogonality to derive (7) for weights $w(x;t)$ such that $w'(x;t)/w(x;t)$ is a rational function;
- Technique was rediscovered by several authors including Bonan, Freud, Mhaskar and Nevai approximately 40 years later;
- Method of ladder operators was introduced by Chen and Ismail [1997];
- Chen and Feigin [2006] adapted the method of ladder operators to the situation where the weight function vanishes at one point.
- Clarkson, J and Kelil [2016] generalize the work by Chen and Feigin, giving a more explicit expression for the coefficients in (7) when the weight function is positive on the real line except for one point.

The differential-difference relation

Theorem (Clarkson, J & Kelil, 2016)

For the generalized Freud weight

$$w(x;t) = |x|^{2\lambda+1} \exp\left(-x^4 + tx^2\right), \qquad x \in \mathbb{R},\ \lambda > 0$$

the monic orthogonal polynomials $S_n(x;t)$ satisfy

$$x \frac{d}{dx} S_n(x;t) = \sum_{j=-1}^{0} a_{n,n+j} S_{n+j}(x;t)$$

with

$$a_{n,n-1} = 4\beta_n x(x^2 - \tfrac{1}{2}t + \beta_n + \beta_{n+1}),$$
$$a_{n,n} = -4x^2\beta_n - \frac{(2\lambda+1)[1-(-1)^n]}{2}.$$

The inverse problem: Given a three-term recurrence relation

Operator theory plays an important role in answering questions concerning

- uniqueness of the orthogonality measure
- nature of the orthogonality measure
- support of the orthogonality measure

The direct problem: Given an orthogonality measure μ

Extracting information from the measure regarding characterising properties of the orthogonal polynomials, such as the coefficients of the

- polynomials
- three-term recurrence relation
- differential-difference equation
- differential equation

For most classical orthogonal polynomials, the properties satisfied are known.

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For **most** **classical orthogonal polynomials**, the properties satisfied are known.

What defines a classical orthogonal polynomial?

A sequence of orthogonal polynomials is classical if the sequence $\{P_n\}$ as well as $D^m P_{n+m}$, $m \in \mathbb{N}$, where D is the usual derivative $\frac{d}{dx}$ or one of its extensions

- difference operator
- q-difference operator
- divided-difference operator

satisfies a three term recurrence of the form ensuring orthogonality by the spectral theorem.

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The following definition of classical orthogonal polynomials, suggested by Andrews and Askey in 1985 is generally accepted and has been justified by various characterisations.

"A set of orthogonal polynomials is classical, if it is a special case or limiting case of the Askey-Wilson polynomials"

Characterizations using structural relations

Consider a structural relation of type

$$\pi(x)SP_n(x) = \sum_{j=-t}^{s} a_{n,n+j}P_{n+j}(x), \quad n = 1, 2, \ldots \tag{7}$$

where $\pi(x)$ is a polynomial and S is a linear operator that maps a polynomial of precise degree n to a polynomial of degree $n-1$.

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Replacing S in (8) by the difference operator $\Delta f(s) = f(s+1) - f(s)$, García, Marcellán and Salto [1995] proved that Hahn, Krawtchouk, Meixner and Charlier polynomials are the only orthogonal polynomial sequences satisfying

$$\pi(x) \Delta P_n(x) = \sum_{j=-1}^{1} a_{n,n+j} P_{n+j}(x), \quad n = 1, 2, \ldots,$$

with $\pi(x)$ a polynomial of degree at most two.

More recently, replacing the operator S in (8) by the Hahn operator

$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)x},$$

Datta and Griffin [2006] characterized the big q-Jacobi polynomial or one of its special or limiting cases (Al-Salam-Carlitz 1, little and big q-Laguerre, little q-Jacobi, and q-Bessel polynomials) as the only orthogonal polynomials that satisfy

$$\pi(x) D_q P_n(x) = \sum_{j=-1}^{1} a_{n,n+j} P_{n+j}, \quad n = 1, 2, \ldots$$

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The polynomials mentioned above are all special or limiting cases of the Askey-Wilson polynomials

$$\frac{a^n p_n(x; a, b, c, d|q)}{(ab, ac, ad; q)_n} = {}_4\phi_3 \left(\begin{matrix} q^{-n}, \; abcdq^{n-1}, \; ae^{-i\theta}, ae^{i\theta} \\ ab, \; ac, \; ad \end{matrix} ; q, q \right), \; x = \cos\theta,$$

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Askey-Wilson polynomials do not satisfy any of these structural relations.

Extension of Askey's problem

Find a structural relation of type

$$\pi(x)\mathcal{S}P_n(x) = \sum_{j=-t}^{s} a_{n,n+j}P_{n+j}(x),$$

that characterises the Askey-Wilson polynomials.

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$\mathcal{D}_q$ is the Askey-Wilson divided difference operator, taking $e^{i\theta} = q^s$,

$$\mathcal{D}_q f(x(s)) = \frac{f(x(s+\frac{1}{2})) - f(x(s-\frac{1}{2}))}{x(s+\frac{1}{2}) - x(s-\frac{1}{2})}, \quad x(s) = \frac{q^{-s} + q^s}{2}$$

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Ismail [2005] gave an important hint to the solution of this problem.

Conjecture (Ismail, 2005)

Let $\{P_n\}$ be orthogonal polynomials and π a polynomial of degree at most 4. Then $\{P_n(x)\}$ satisfies

$$\pi(x)\mathcal{D}_q^2 P_n(x) = \sum_{j=-t}^{s} a_{n,j} P_{n+j}(x)$$

if and only if $\{P_n(x)\}$ are Askey-Wilson polynomials or special cases of them.

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$$S = \mathcal{D}_q^2$$

The aim of what follows is to complete and prove the above conjecture.

Characterizations using second order operator equations

When $D = \frac{d}{dx}$, Hahn [1936] showed that a sequence of monic orthogonal polynomials $\{P_n(x)\}_{n=0}^{\infty}$ satisfying

$$\frac{1}{n+1}\frac{dP_{n+1}}{dx}(x) = (x - \widetilde{a}_n)\frac{1}{n}\frac{dP_n}{dx}(x) - \frac{\widetilde{b}_n}{n-1}\frac{dP_{n-1}}{dx}(x), \quad \widetilde{a}_n, \widetilde{b}_n \in \mathbb{R}, \widetilde{b}_n \neq 0,$$

satisfies a second order Sturm-Liouville differential equation of the form

$$\phi(x)\frac{d^2}{dx^2}P_n(x) + \psi(x)\frac{d}{dx}P_n(x) + \lambda_n P_n = 0. \tag{8}$$

where, ϕ and ψ are polynomials independant of n with $\deg(\phi) \leq 2$ and $\deg(\psi) = 1$ while λ_n is a constant dependant on n.

Bochner [1929] first considered sequences of polynomials satisfying (9) and showed that the orthogonal polynomial solutions of (9) are Jacobi, Laguerre and Hermite polynomials.

Ismail [2003] generalized Bochner's theorem to Askey-Wilson divided difference operators. A more general version is due to Vinet and Zhedanov [2008].

Lemma (Kenfack-Nangho & J)

Let $\{P_n\}_{n=0}^{\infty}$ a sequence of monic orthogonal polynomials. If there are two sequences (a'_n) and (b'_n) of numbers such that

$$\frac{1}{\gamma_{n+1}}\mathcal{D}_q P_{n+1}(x) = (x - a'_n)\frac{1}{\gamma_n}\mathcal{D}_q P_n(x) - \frac{b'_n}{\gamma_{n-1}}\mathcal{D}_q P_{n-1}(x) + c_n, \;\; c_n \in \mathbb{R},$$

then, there are two polynomials $\phi(x)$ and $\psi(x)$ of degree at most two and of degree one respectively and a sequence $\{\lambda_n\}_{n=0}^{\infty}$ depending on n such that $P_n(x)$ satisfies the divided-difference equation

$$\phi(x)\mathcal{D}_q^2 P_n(x) + \psi(x)\mathcal{S}_q\mathcal{D}_q P_n(x) + \lambda_n P_n(x) = 0, \;\; n \geq 5.$$

Lemma (Kenfack-Nangho & J)

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$$\phi(x)\mathcal{D}_q^2 P_n(x) + \psi(x)\mathcal{S}_q\mathcal{D}_q P_n(x) + \lambda_n P_n(x) = 0, \ n \geq 5.$$

Here $\mathcal{S}_q$ is the averaging operator

$$\mathcal{S}_q f(x(s)) = \frac{f(x(s+\frac{1}{2})) + f(x(s-\frac{1}{2}))}{2}.$$

Theorem (Kenfack-Nangho & J)

Let $\{P_n\}_{n=0}^{\infty}$ be a sequence of polynomials orthogonal with respect to a positive weight function $w(x)$. The following properties are equivalent

a) *There is a polynomial $\pi(x)$ of degree at most 4 and constants $a_{n,n+j}$, $-2 \leq j \leq 2$, $n \geq 2$, $a_{n,n-2} \neq 0$ such that P_n satisfies the structure relation*

$$\pi(x)\mathcal{D}_q^2 P_n(x) = \sum_{j=-2}^{2} a_{n,n+j} P_{n+j}(x);$$

b) *There is a polynomial $\pi(x)$ of degree at most four such that $\{\mathcal{D}_q^2 P_n\}_{n=2}^{\infty}$ is orthogonal with respect to $\pi(x)\, w(x)$;*

c) *There are two polynomials $\phi(x)$ and $\psi(x)$ of degree at most two and of degree one respectively and a constant λ_n such that*

$$\phi(x)\mathcal{D}_q^2 P_n(x) + \psi(x)\mathcal{S}_q\mathcal{D}_q P_n(x) + \lambda_n P_n(x) = 0, \quad n = 5, 6, \ldots.$$

Corollary (Kenfack-Nangho & J)

A sequence of monic orthogonal polynomials satisfies the relation

$$\pi(x)\mathcal{D}_q^2 P_n(x) = \sum_{j=-2}^{2} a_{n,n+j} P_{n+j}(x), \quad a_{n,n-2} \neq 0, x = \cos\theta,$$

where π is a polynomial of degree at most 4, if and only if $P_n(x)$ is a multiple of the Askey-Wilson polynomial for some parameters a, b, c, d, including limiting cases as one or more of the parameters tends to ∞.

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Limiting cases?

- continuous dual q-Hahn polynomials;
- Al-Salam-Chihara polynomials;
- continuous big q-Hermite polynomials;
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Missing cases?

An application

$$\pi(x)\mathcal{D}_q^2 P_n(x) = \sum_{j=-2}^{2} a_{n,n+j} P_{n+j}(x), \quad a_{n,n-2} \neq 0, x = \cos\theta,$$

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Theorem

Let $\{p_n\}_{n=0}^{\infty}$ be a sequence of polynomials orthogonal on the interval (c, d). Fix $k, n \in$ with $k < n-1$ and suppose deg(g_{n-k-1})$= n-k-1$ with

$$\pi(x)g_{n-k-1}(x) = G_k(x)p_{n-1}(x) + H(x)p_n(x)$$

where $\pi(x) \neq 0$ for $x \in (c, d)$ and $deg(G_k) = k$.

Then the largest (smallest) zero of G_k is a strict lower (upper) bound for the largest (smallest) zero of p_n.

Proposition (Kenfack-Nangho & J)

Let $n \in \mathbb{N}$ be fixed and $(x_{n,n})$ be the largest zero of the monic Askey-Wilson polynomial $P_n(x, a, b, c, d|q)$. Then a lower bound for $x_{n,n}$ is

$$\frac{2(q^{n-1}+1)\left(Aq^{n-1}-a-b-c-d\right)\left(abcdq^{n-1}-1\right)+\sqrt{D_n}}{8\left(abcdq^{2n-2}-1\right)\left(abcdq^{n-1}-1\right)}$$

where

$$A = (abc + abd + acd + bcd)$$

and

$$\begin{aligned}
D_n = {} & -(4\left(-q^{3n-3}abcd-1\right)(abcd-ab-ac-ad-bc-bd-cd+1) \\
& +4\left((b^2c^2d^2+b^2c^2+b^2cd+b^2d^2+bc^2d+bcd^2+c^2d^2-bc-bd-cd)a^2\right. \\
& \left.+(bc+bd+cd)(bcd-b-c-d)a+bcd(bcd-b-c-d)\right)q^{2n-2} \\
& +4\left((-bc-bd-cd+1)a^2-(bc+bd+cd-1)(d+c+b)a-b^2cd-bc^2d-bcd^2\right. \\
& \left.+b^2+bc+bd+c^2+cd+d^2+1)q^{n-1}\right)\left(4\,abcdq^{2n-2}-4\right)\left(abcdq^{n-1}-1\right) \\
& +4\left(q^{n-1}+1\right)^2\left(q^{n-1}abc+q^{n-1}abd+q^{n-1}acd+q^{n-1}bcd-a-b-c-d\right)^2\left(abcdq^{n-1}-1\right)^2
\end{aligned}$$

The zeros of monic Askey-Wilson polynomials $P_n(x; a, b, c, d|q)$ and the bounds obtained from the structural relation.

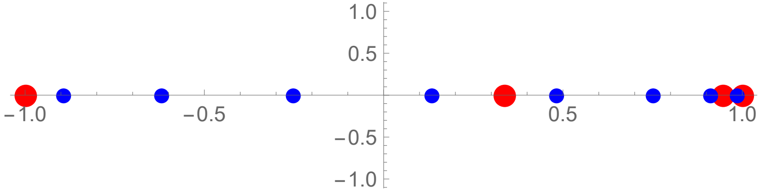

Figure: Plots of zeros of $P_n(x; a, b, c, d|q)$ (blue) for $n = 8$, $a = \frac{6}{7}$, $b = \frac{5}{7}$, $c = \frac{4}{7}$, $d = \frac{3}{7}$, $q = \frac{1}{9}$ and the bounds for the extreme zeros (red).

THANK YOU FOR YOUR ATTENTION!

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