Zeros of orthogonal polynomials

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Douala, Cameroon

Outline

Jacobi matrix

Let $\{p_n(x)\}_{n=0}^{\infty}$ be a sequence of monic orthogonal polynomials satisfying

$$p_{n+1}(x) = (x - \alpha_n)p_n(x) - \beta_n \, p_{n-1}(x), \quad n = 0, 1, 2, \ldots,$$

with $p_{-1} = 0$ and $p_0 = 1$.

The recurrence coefficients can be collected in a tridiagonal matrix of the form

$$J = \begin{pmatrix} \alpha_0 & \sqrt{\beta_1} & & & \\ \sqrt{\beta_1} & \alpha_1 & \sqrt{\beta_2} & & \\ & \sqrt{\beta_2} & \alpha_2 & \sqrt{\beta_3} & \\ & & \sqrt{\beta_3} & \alpha_3 & \ddots \\ & & & \ddots & \ddots \end{pmatrix}$$

known as the Jacobi matrix.

Zeros as eigenvalues

One can write

$$p_n(x) = \det\left(xI_n - J_n\right)$$

where I_n is the identity matrix and J_n is the tridiagonal matrix

$$J_n = \begin{pmatrix} \alpha_0 & \sqrt{\beta_1} & & & & \\ \sqrt{\beta_1} & \alpha_1 & \sqrt{\beta_2} & & & \\ & \sqrt{\beta_2} & \alpha_2 & \sqrt{\beta_3} & & \\ & & \sqrt{\beta_3} & \alpha_3 & \ddots & \\ & & & \ddots & \ddots & \\ & & & & & \sqrt{\beta_{n-1}} \\ & & & & \sqrt{\beta_{n-1}} & \alpha_{n-1} \end{pmatrix}$$

It follows that zeros of $p_n(x)$ are the same as the eigenvalues of J_n.

Real and distinct zeros on interval of orthogonality

Theorem

If $\{p_n(x)\}_{n=0}^{\infty}$ is a sequence of orthogonal polynomials on the interval $(a,\ b)$ with respect to the weight function $w(x)$, then the polynomial $p_n(x)$ has exactly n real simple zeros in the interval $(a,\ b)$.

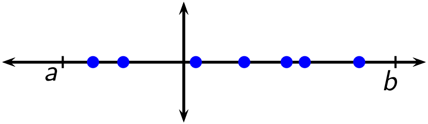


Figure: Zeros in interval I

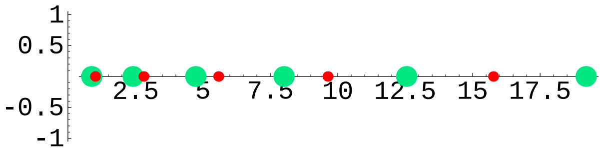
Interlacing of zeros

Theorem

If $\{p_n(x)\}_{n=0}^{\infty}$ is a sequence of orthogonal polynomial on the interval (a, b) with respect to the weight function $w(x)$, then the zeros of $p_n(x)$ and $p_{n+1}(x)$ separate each other.

If $\{x_{n,k}\}_{k=1}^{n}$ and $\{x_{n+1,k}\}_{k=1}^{n+1}$ denote the consecutive zeros of $p_n(x)$ and $p_{n+1}(x)$ respectively, then we have

$$a < x_{n+1,1} < x_{n,1} < x_{n+1,2} < \ldots < x_{n+1,n} < x_{n,n} < x_{n+1,n+1} < b.$$



Monotonicity of the zeros

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Markov's approach works in a more general framework.

The monotonicity theorem due to Markov

Theorem (cf. (Ismail, Theorem 6.21.1, p.121))

Let $\{p_n(x,\tau)\}_{n=0}^{\infty}$ *be orthogonal with respect to* $d\alpha(x,\tau) = w(x,\tau)d\alpha(x)$ *on the interval* $[a,b]$ *depending on a parameter* τ *such that* $w(x,\tau)$ *is positive and continuous for* $x \in (a,b)$, $\tau \in (\tau_1,\tau_2)$.

Suppose that $w_\tau(x,\tau)$ *exists and is continuous, and the integrals*

$$\int_a^b x^k w_\tau(x,\tau)d\alpha(x), \ \ k = 0,1,2,\dots,2n-1,$$

converge uniformly in every closed subinterval of (τ_1,τ_2).

Denote the zeros of $p_n(x,\tau)$ *by* $x_1(\tau) < x_2(\tau) < \cdots < x_n(\tau)$. *Then the* k*th zero* $x_k(\tau)$ *is an increasing function of* τ *provided that*

$$w_\tau / w$$

is an increasing function of $x \in (a,b)$.

Proof of the monotonicity theorem

The mechanical quadrature formula

$$\int_a^b \rho(x)d\alpha(x,\tau) = \sum_{k=1}^{n} \lambda_k(\tau)\rho(x_k(\tau)),$$

holds for polynomials $\rho(x)$ of degree at most $2n-1$.
Differentiating with respect to τ, we obtain

$$\int_a^b \rho(x)w_\tau(x,\tau)d\alpha(x) = \sum_{k=1}^{n} \lambda_k(\tau)\rho'(x_k)x_k'(\tau) + \sum_{k=1}^{n} \lambda_k'(\tau)\rho(x_k).$$

Now we choose

$$\rho(x) = \frac{\{p_n(x,\tau)\}^2}{x-x_k},$$

then, since x_k is a removable singularity, $\rho'(x_k) = \{p_n'(x_k,\tau)\}^2$ while $\rho'(x_\ell) = 0$ if $\ell \neq k$ and hence

$$\int_a^b \frac{\{p_n(x,\tau)\}^2}{x-x_k} w_\tau(x,\tau)d\alpha(x) = \lambda_k(\tau)\{p_n'(x_k,\tau)\}^2 x_k'(\tau).$$

$$\int_a^b \frac{\{p_n(x,\tau)\}^2}{x - x_k} w_\tau(x,\tau) d\alpha(x) = \lambda_k(\tau)\{p_n^{'}(x_k,\tau)\}^2 x_k^{'}(\tau).$$

$$\int_a^b \frac{\{p_n(x,\tau)\}^2}{x-x_k} w_\tau(x,\tau)d\alpha(x) = \lambda_k(\tau)\{p_n^{'}(x_k,\tau)\}^2 x_k^{'}(\tau).$$

In view of the orthogonality the integral

$$\int_a^b \frac{\{p_n(x,\tau)\}^2}{x-x_k} w(x,\tau)d\alpha(x) = 0,$$

we can rewrite this as

$$\int_a^b \frac{\{p_n(x,\tau)\}^2}{x-x_k} \left\{ \frac{w_\tau(x,\tau)}{w(x,\tau)} - \frac{w_\tau(x_k,\tau)}{w(x_k,\tau)} \right\} w(x,\tau)d\alpha(x) \\ = \lambda_k(\tau)\{p_n^{'}(x_k,\tau)\}^2 x_k^{'}(\tau).$$

The integrand has a constant sign, so the positivity of the so-called Christoffel numbers $\lambda_k(\tau)$ (cf. [Szegö, p. 48] establishes the result.

Jacobi polynomials

Example

For $P_n^{(\alpha,\beta)}$, $w(x,\alpha,\beta) = (1-x)^\alpha(1+x)^\beta$ and $\alpha(x) = x$, hence

$$\frac{\partial \ln w(x,\alpha,\beta)}{\partial \beta} = \frac{\partial \ln (1-x)^\alpha(1+x)^\beta}{\partial \beta} = \ln(1+x), \; x \in (-1,1)$$

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Lemma (cf. (Szego, Theorem 6.21.1, p.121))

Let $\alpha, \beta > -1$ and let x_k, $k = 1, 2, \ldots, n$ denote the zeros of $P_n^{(\alpha,\beta)}$. Then $\frac{dx_k}{d\alpha} < 0$ and $\frac{dx_k}{d\beta} > 0$ for each fixed k.

Zeros of $P_n^{(\alpha,\beta)}$

- increase as β increases
- decrease as α increases.

Electrostatic interpretations of the zeros of the classical orthogonal polynomials originally given by Stieltjes [1885] and generalised by Ismail [2000] imply that increasing a parameter corresponds to increasing a charge at one endpoint.

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- Fix two positive charges of magnitude $\frac{\beta+1}{2}$ at -1, $\frac{\alpha+1}{2}$ at $+1$
- n movable positive unit charges move freely in $(-1, 1)$
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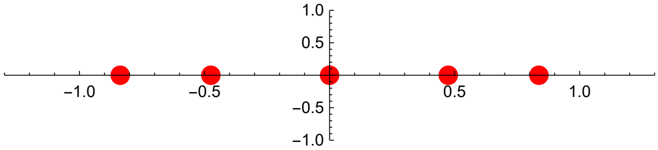

Figure: Zeros of $P_5^{(0.9,0.9)}(x)$.The charges at the endpoints are both 0.95.

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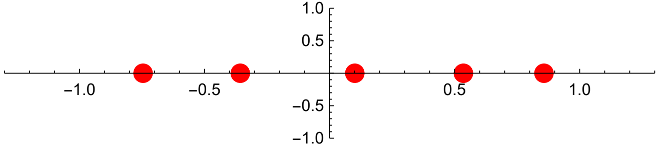

Figure: Zeros of $P_5^{(0.9,1.9)}(x)$. The charge at -1 is 1.45 and at 1 is 0.95.

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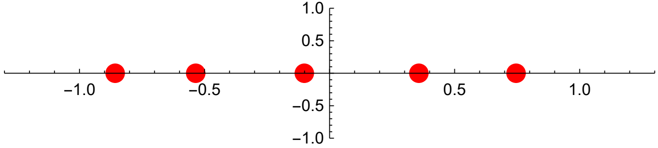

Figure: Zeros of $P_5^{(1.9,0.9)}(x)$. The charge at -1 is 0.95 and at 1 is 1.45.

Interlacing of zeros from different sequences

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Theorem (Driver and J, 2007)

The zeros of $L_n^{\alpha}(x)$ *and* $L_n^{\alpha+t}(x)$ *interlace for* $\alpha > -1$ *and* $t \in (0,2]$.

Laguerre polynomials for $\alpha > -1$

$$
\begin{array}{ccccc}
L_1^{\alpha} & L_1^{\alpha+1} & L_1^{\alpha+2} & L_1^{\alpha+3} & \cdots \\
L_2^{\alpha} & L_2^{\alpha+1} & L_2^{\alpha+2} & L_2^{\alpha+3} & \cdots \\
L_3^{\alpha} & L_3^{\alpha+1} & L_3^{\alpha+2} & L_3^{\alpha+3} & \cdots \\
L_4^{\alpha} & L_4^{\alpha+1} & L_4^{\alpha+2} & L_4^{\alpha+3} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \\
L_{n-1}^{\alpha} & L_{n-1}^{\alpha+1} & L_{n-1}^{\alpha+2} & L_{n-1}^{\alpha+3} & \cdots \\
L_n^{\alpha} & L_n^{\alpha+1} & L_n^{\alpha+2} & L_n^{\alpha+3} & \cdots \\
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\vdots & \vdots & \vdots & \vdots & \\
\updownarrow\, L_{n-1}^{\alpha} & \updownarrow\, L_{n-1}^{\alpha+1} & \updownarrow\, L_{n-1}^{\alpha+2} & \updownarrow\, L_{n-1}^{\alpha+3} & \cdots \\
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L_{n-1}^{\alpha} & L_{n-1}^{\alpha+1} & L_{n-1}^{\alpha+2} & L_{n-1}^{\alpha+3} & \cdots \\
L_n^{\alpha} \longleftrightarrow & L_n^{\alpha+1} & L_n^{\alpha+2} & L_n^{\alpha+3} & \cdots \\
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Interlacing of zeros of $P_n^{(\alpha,\beta)}$ and $P_{n-1}^{(\alpha',\beta')}$ $\alpha, \beta > -1$

Theorem (Driver, J and Mbuyi, 2008)

Let $\alpha,\ \beta > -1$ *and let* $0 \leq t \leq 2$ *and* $0 \leq k \leq 2$. *Let*

$$\begin{array}{ll} -1 < x_1 < x_2 < \cdots < x_n < 1 & \text{be the zeros of } P_n^{(\alpha,\beta)} \text{ and} \\ -1 < t_1 < t_2 < \cdots < t_{n-1} < 1 & \text{be the zeros of } P_{n-1}^{(\alpha+t,\beta+k)}. \end{array}$$

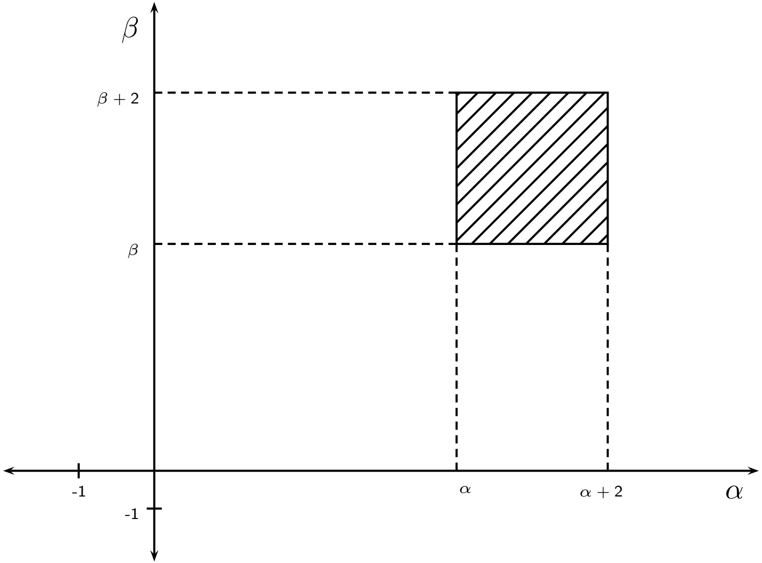
Then

$$-1 < x_1 < t_1 < x_2 < \cdots < x_{n-1} < t_{n-1} < x_n < 1.$$

Remark

Some restrictions on the ranges of t *and* k *are required in the theorems since the interlacing property is not retained, in general, when one or both of the parameters* α, β *are increased by more than* 2.

Interlacing of zeros of $P_n^{(\alpha,\beta)}$ and $P_{n-1}^{(\alpha',\beta')}$, $\alpha, \beta, \alpha', \beta' > -1$



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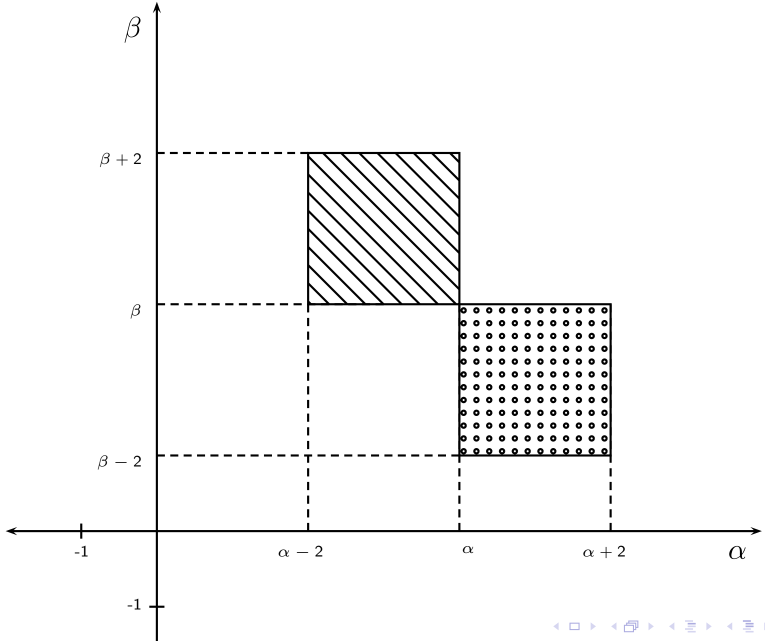

Stieltjes interlacing

Theorem

Let $\{p_n\}_{n=0}^{\infty}$ be a sequence of orthogonal polynomials on the interval (a, b) with respect to $w(x) > 0$ and suppose $m < n$. Then, between any two zeros of p_m, there is at least one zero of p_n.

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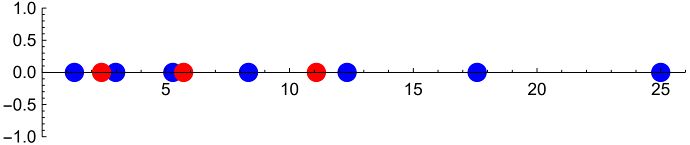

Figure: Zeros of $L_7^{\alpha}(x)$ (blue) and $L_3^{\alpha}(x)$ (red) for $\alpha = 3.4$

Proof

Suppose that $x_{m,k}$ and $x_{m,k+1}$ are two consecutive zeros of $p_m(x)$ and that there is no zero of $p_n(x)$ in $(x_{m,k}, x_{m,k+1})$. Consider

$$g(x) = \frac{p_m(x)}{(x - x_{m,k})(x - x_{m,k+1})}.$$

Then $g(x)p_m(x) \geq 0$ for $x \neq (x_{m,k}, x_{m,k+1})$.

If $\{x_{n,i}\}_{i=1}^{n}$ are the zeros of $p_n(x)$, Gauss quadrature gives

$$\int_a^b g(x)p_m(x)w(x)dx = \sum_{i=1}^{n} \lambda_{n,i} g(x_{n,i}) p_m(x_{n,i}).$$

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Since there are no zeros of $p_n(x)$ in $(x_{m,k}, x_{m,k+1})$ we conclude that $g(x_{n,i})p_m(x_{n,i}) \geq 0$ for all $i = 1, 2, \ldots, n$. Further we have $\lambda_{n,i} > 0$ for all $i = 1, 2, \ldots, n$ which implies that the sum on the right-hand side cannot vanish. However, the integral on the left-hand side is zero by orthogonality and we have a contradiction.

Bounds for the zeros of orthogonal polynomials

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- difference equations;
- Sturmian methods for differential equation;
- Obrechkov's theorem;
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Mixed recurrence relations used to prove Stieltjes interlacing of the zeros of two polynomials from different sequences provide a set of points that can be applied as inner bounds for the extreme zeros of polynomials.

Bounds following from Stieltjes interlacing

Theorem (Driver & J, 2012)

Let $\{p_n\}_{n=0}^{\infty}$ be a sequence of polynomials orthogonal on the interval (c, d). Fix $k, n \in \mathbb{N}$ with $k < n - 1$ and suppose deg(g_{n-k-1})$= n - k - 1$ with

$$f(x)g_{n-k-1}(x) = G_k(x)p_{n-1}(x) + H(x)p_n(x) \qquad (1)$$

where $f(x) \neq 0$ for $x \in (c, d)$ and $deg(G_k) = k$.

Then the $n - 1$ real, simple zeros of $G_k g_{n-k-1}$ interlace with the zeros of p_n if g_{n-k-1} and p_n are co-prime.

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where $f(x) \neq 0$ *for* $x \in (c,d)$ *and* $deg(G_k) = k$*.*

Then the $n-1$ *real, simple zeros of* $G_k g_{n-k-1}$ *interlace with the zeros of* p_n *if* g_{n-k-1} *and* p_n *are co-prime.*

Corollary

Suppose (1) holds for $k,n \in \mathbb{N}$ *fixed and* $k < n-1$*. The largest (smallest) zero of* G_k *is a strict lower (upper) bound for the largest (smallest) zero of* p_n*.*

Proof

Let $w_1 < \cdots < w_n$ denote the zeros of p_n.

p_{n-1} and p_n are always co-prime while p_n and g_{n-k-1} are co-prime by assumption, so it follows from (1) that $G_k(w_j) \neq 0$ for every j.

From (1), provided $p_n(x) \neq 0$, we have

$$\frac{f(x)g_{n-k-1}(x)}{p_n(x)} = H(x) + \frac{G_k(x)p_{n-1}(x)}{p_n(x)}.$$

The decomposition into partial fractions [Szegö, Theorem 3.3.5]

$$\frac{p_{n-1}(x)}{p_n(x)} = \sum_{j=1}^{n} \frac{A_j}{x - w_j},$$

where $A_j > 0$ for every $j \in \{1, \ldots, n\}$, implies that we can write

$$\frac{f(x)g_{n-k-1}(x)}{p_n(x)} = H(x) + \sum_{j=1}^{n} \frac{G_k(x)A_j}{x - w_j}, \quad x \neq w_j.$$

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Suppose that G_k does not change sign in an interval (w_j, w_{j+1}) where $j \in \{1, 2, \ldots, n-1\}$.

Since $A_j > 0$ and the polynomial H is bounded on I_j while the right hand side takes arbitrarily large positive and negative values on (w_j, w_{j+1}), it follows that g_{n-k-1} must have an odd number of zeros in every interval in which G_k does not change sign.

Since G_k is of degree k, there are at least $n-k-1$ intervals (w_j, w_{j+1}), $j \in \{1, \ldots, n-1\}$ in which G_k does not change sign and so each of these intervals must contain exactly one of the $n-k-1$ real, simple zeros of g_{n-k}.

We deduce that the k zeros of G_k are real and simple and, together with the $n-k-1$ zeros of g_{n-k-1}, interlace with the n zeros of p_n.

Example

For Jacobi polynomials $P_n^{(a,\beta)}$, $\alpha, \beta > -1$, one can show that, when $n > 1$, $n \in \mathbb{N}$, (1) holds for $k = 1$ with

$$g_{n-1} = P_{n-2}^{(\alpha+4,\beta)}$$

$$G_1(x) = x - \frac{2(n-1)(n+\alpha+\beta+2) + (\alpha+3)(\beta-\alpha)}{2(n-1)(n+\alpha+\beta+2) + (\alpha+3)(\alpha+\beta+2)}$$

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It follows that for all $\alpha, \beta > -1$, $n \in \mathbb{N}$,

$$w_n > 1 - \frac{2(\alpha+1)(\alpha+3)}{2(n-1)(n+\alpha+\beta+2) + (\alpha+3)(\alpha+\beta+2)}$$
$$= 1 - O(\frac{1}{n^2})$$

This bound is sharper than the lower bound for largest zero [Szegö]

$$1 - \frac{2(\alpha+1)}{2n+\alpha+\beta} = 1 - O(\frac{1}{n})$$

Sturm's convexity Theorem

$$y''(t) + F(t)y(t) = 0$$

a second-order differential equation in normal form, where F is continuous in $(a,\ b)$.

$y(t)$ – a nontrivial solution in $(a,\ b)$, and

$x_1 < \ldots < x_k < x_{k+1} < \ldots$ – the consecutive zeros of $y(t)$.

Theorem

- *If $F(t)$ is strictly increasing in $(a,\ b)$, then $x_{k+2} - x_{k+1} < x_{k+1} - x_k$ (zeros are concave).*
- *If $F(t)$ is strictly decreasing in $(a,\ b)$, then $x_{k+2} - x_{k+1} > x_{k+1} - x_k$ (zeros are convex).*

Sturm's Theorem

Moreover,

- *if there exists* $M > 0$ *such that* $F(t) < M$ *in* $(a,\ b)$ *then*

$$\Delta x_k \equiv x_{k+1} - x_k > \frac{\pi}{\sqrt{M}},$$

- *if there exists* $m > 0$ *such that* $F(t) > m$ *in* $(a,\ b)$ *then*

$$\Delta x_k < \frac{\pi}{\sqrt{m}}.$$

Transforming the zeros

Problem: The second-order differential equations for orthogonal polynomials and special functions are not in normal form!
Solution:

- Szegő (1938): *If* $-1/2 < \alpha = \beta < 1/2$*, then the sequence*

$$\theta_0,\ \theta_1,\ \theta_2,\ \dots\ ,\theta_{[n/2]+1}$$

of the zeros of $P_n^{(\alpha,\ \alpha)}(\cos\theta)$ *is convex.*

- Deano, Gil, Segura (2004) – Applying Liouville transformation to obtain information on the convexity of the transformed zeros for hypergeometric functions.

Easiest transformation

$$x'' + g(t)x' + f(t)x = 0$$

With the transformation

$$y = x \exp\left(\frac{1}{2}\int^t g(s)ds\right)$$

the normal form is

$$y'' + F(t)y = 0,$$

where $F(t) = f(t) - \frac{1}{4}g^2(t) - \frac{1}{2}g'(t)$.

Advantage: the zeros of x and y are the same!

- Sturm (1836) used it for Bessel functions
- Hille (1933) used it for Hermite polynomials

Laguerre polynomials

The Laguerre polynomials are orthogonal on $(0, \infty)$ when $\alpha > -1$. Differential equation:

$$tx'' + (\alpha + 1 - t)x' + nx = 0.$$

Normal form:

$$y'' + F(t)y = 0,$$

where

$$F(t) = \frac{-t^2 + 2\alpha t + 2t + 4nt - \alpha^2 + 1}{4t^2}.$$

$F(t)$ changes monotonicity at

$$t_0 := \frac{\alpha^2 - 1}{\alpha + 2n + 1}.$$

Theorem (Toókos, J, 2009)

The zeros of $L_n^\alpha(t)$ on $(0, \infty)$ are

1. *all convex if $n > 0$ and $-1 < \alpha \leq 3$*
2. *all convex if $\alpha > 3$ and $0 < n < \frac{\alpha+1}{\alpha-3}$*
3. *concave for $t < t_0$ and convex for $t > t_0$ when $\alpha > 3$ and $n > \frac{\alpha+1}{\alpha-3}$.*

Moreover, for the distance between consecutive zeros we have

$$\Delta x_k > \frac{\pi\sqrt{2}}{\sqrt{2\alpha n + \alpha + 2n^2 + 2n + 1}} \qquad k = 1, \ldots, n-1,$$

and also if $x_k > t_0$ then

$$\frac{\pi}{\sqrt{F(x_k)}} < \Delta x_k < \frac{\pi}{\sqrt{F(x_{k+1})}} \qquad k = 1, \ldots, n-2.$$

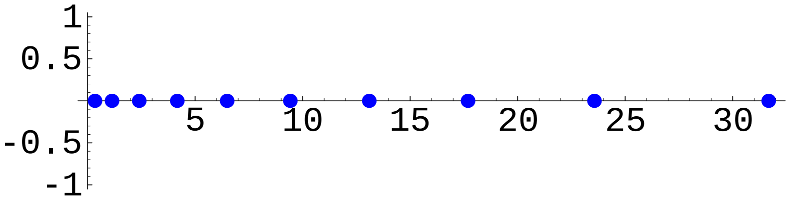

Figure: Zeros of the Laguerre polynomial for $\alpha = 0.98887$ and $n = 10$

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- $t_0 < \dfrac{(\alpha+1)(\alpha+2)}{\alpha+n+1}$, *upper bound for x_1 given by Hahn.*
- $t_0 < \dfrac{(\alpha+1)(\alpha+3)}{\alpha+2n+1}$, *upper bound for x_1 given by Szegő.*

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- $2n+\alpha-2-\sqrt{1+(n-1)(n+\alpha-1)}$, Driver & Jordaan?