Lab. Physics-Mathematica, Special Function and Applications
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Linearization coefficient for some basic hypergeometric polynomials.

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Content

Basic hypergeometric functions

Definition

Let a and q be two complex numbers:

$$\lim_{q\to 1}\frac{1-q^a}{1-q}=a.$$

$$(a;q)_n=\begin{cases}(1-a)(1-aq)\cdots(1-aq^{n-1}) & \text{if} \quad n\in\mathbb{N}^*\\ 1 & \text{if} \quad n=0.\end{cases} \tag{1}$$

$$\lim_{q\to 1^-}\frac{(q^a;q)_k}{(1-q)^k}=(a)_k:=\prod_{r=0}^{k-1}(a+r). \tag{2}$$

$$(a,q)_\infty=\prod_{n=0}^{\infty}(1-aq^n)\text{ if }|q|<1. \tag{3}$$

$$(a_1,a_2,\cdots,a_p,q)_n=(a_1,q)_n\cdots(a_p,q)_n. \tag{4}$$

Basic hypergeometric functions

Definition (Basic hypergeometric functions)

The basic hypergeometric functions ${}_r\phi_s$ is defined by:

$$ {}_r\phi_s\left(\begin{array}{c}(a_r)\\(b_s)\end{array};q;z\right)=\sum_{n=0}^{\infty}\frac{(a_1,...,a_r;q)_n}{(q;q)_n(b_1,...,b_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n. \quad (5) $$

Basic Kampé de Fériet function

Definition (Srivastava and Karlsson ,1985)

The basic Kampé de Fériet function is a generalized bivariate basic hypergeometric function as follows

$$\phi^{p:s;r}_{t:u;v}\left[\begin{array}{ccccccccccc}(a_p) & : & (b_s) & ; & (c_r) & ; & q & : & x & , & y\\ (\alpha_t) & : & (\beta_u) & ; & (\gamma_v) & ; & i & , & j & , & k\end{array}\right]$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{[a_p;q]_{m+n}[b_s;q]_m[c_r;q]_n}{[\alpha_t;q]_{m+n}[\beta_u;q]_m[\gamma_v;q]_n}\frac{x^m y^n q^{i\binom{m}{2}+j\binom{n}{2}+kmn}}{(q;q)_m(q;q)_n}. \quad (6)$$

Linearization problems

The linearization problem consists in finding the coefficients $L_{ij}(k)$ in the expression:

$$P_i(x)Q_j(x) = \sum_{k=0}^{i+j} L_{ij}(k)R_k(x), \quad (7)$$

with $\{P_n\}_{n\geq 0}, \{Q_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ three polynomial sequences. In the special case, $P_n = Q_n = R_n$ the problem is reduced to the standard linearization or Clebsh-Gordan type problem.

Linearization problem

We are interested in deriving the linearization coefficients for a general class of basic polynomials

$$P_n(x; a, \lambda|q) = \sum_{k=0}^{n} (q^{-n}, aq^n; q)_k \lambda_k q^k x^k \tag{8}$$

The polynomial set defined by (8) contains the basic hypergeometric polynomials given by:

$${}_2\phi_1\left(\begin{matrix} q^{-n}, aq^n \\ b \end{matrix}; q; qx\right), \qquad {}_1\phi_1\left(\begin{matrix} q^{-n} \\ b \end{matrix}; q; qx\right). \tag{9}$$

Linearization problem

To find the coefficients of linearization in (7), we combine the inversion relation

$$x^n = \sum_{k=0}^{n} I_k(n) R_k(x),$$

with the expressions

$$P_i(x) = \sum_{r=0}^{i} A_r(i) x^r, \quad Q_j(x) = \sum_{s=0}^{j} D_s(j) x^s,$$

which yields, with sum manipulation, to the representation

$$L_{ij}(k) = \sum_{r=0}^{i} \sum_{s=0}^{j} A_{i-r}(i) D_{j-s}(j) I_k(i+j-r-s). \tag{10}$$

Theorem

The linearization coefficients for the basic polynomials (8) is given by

$$P_i(x;a,\lambda|q)P_j(x;\alpha,\mu|q)=\sum_{k=0}^{i+j}L_{ij}(k)P_k(x;c,\nu|q), \tag{11}$$

with

$$L_{ij}(k)=\sum_{r=0}^{i}\sum_{s=0}^{j}\frac{(-1)^k q^{\frac{k(k-1)}{2}}(q^{-i},aq^i;q)_{i-r}(q^{-j},\alpha q^j;q)_{j-s}}{(cq^k,q;q)_k(cq^{2k+1},q;q)_{i+j-r-s-k}}\frac{\lambda_{i-r}\mu_{j-s}}{\nu_{i+j-r-s}}q^{i+j-r-s}. \tag{12}$$

Linearization coefficients

Corollary

The linearization formula for basic hypergeometric polynomials (9) is given by

$$ {}_2\phi_1\left(\begin{matrix} q^{-i}, aq^i \\ b \end{matrix}; q; qx\right) {}_2\phi_1\left(\begin{matrix} q^{-j}, \alpha q^j \\ \beta \end{matrix}; q; qx\right) = \sum_{k=0}^{i+j} L_{ij}(k) {}_2\phi_1\left(\begin{matrix} q^{-k}, cq^k \\ d \end{matrix}; q; qx\right), \quad (13) $$

with

$$ L_{ij}(i+j-k) = h_{i,j,k,q} \times \phi^{2:2;2}_{2:1;1}. $$

The linearization coefficients of the basis $(x \ominus a)_q^n$

Definition

The basis $(x \ominus a)_q^n$ is defined by:

$$(x \ominus a)_q^n = (x-a)(x-aq)\dots(x-aq^{n-1}) = (-a)^n q^{\frac{n(n-1)}{2}} {}_1\phi_0\left(\begin{matrix} q^{-n} \\ - \end{matrix}; q; \frac{qx}{a}\right),$$

Proposition

The linearization coefficients of the basis $(x \ominus a)_q^n$:

$$(x \ominus a)_q^i (x \ominus a)_q^j = \sum_{k=0}^{\min(i,j)} (-a)^k q^{\frac{k(k-1)}{2}} \frac{(q;q)_i (q;q)_j}{(q;q)_k (q;q)_{j-k} (q;q)_{i-k}} (x \ominus a)_q^{i+j-k} \tag{14}$$

The linearization coefficients of the basis $(x \ominus a)_q^n$

proof:

By applying Theorem 2.2 for the basis $(x \ominus a)_q^n$ to obtain the linearization formula:

$$(x \ominus a)_q^i (x \ominus a)_q^j = \sum_{k=0}^{i+j} L_{ij}(k)(x \ominus a)_q^k \qquad (15)$$

where

$$L_{ij}(i+j-k) = a^k \begin{bmatrix} i+j \\ k \end{bmatrix}_q \times \phi^{1:1;1}_{1:0;0},$$

Using the explicit formula (16) and the *q-Multisum* package we find the recurrence relation satisfied by $S(k) := L_{ij}(i+j-k)$:

$$a^2 q \left(q^i - q^k\right)\left(q^j - q^k\right)\left(q^{i+j} - q^k\right) S(k) - \left(q^{k+2} - 1\right) q^{2k+1} S(k+2)$$
$$-aq^k \left(q^{i+j+k+2} - q^{i+j} - q^{i+j+1} + q^{i+k+1} + q^{j+k+1} + q^{k+1} - 2q^{2k+2}\right) S(k+1) = 0. \qquad (16)$$

Then by means of *q*-Zeilberger algorithm , via the Maple software, and in particular, the *qrecsolve* command of the *qsum17* package we successfully obtained the exact solution of the recurrence relation (16)

$$S(k) = \frac{\left(q^{-j};q\right)_k \left(q^{-i};q\right)_k \left(-aq^{i+j}\right)^k q^{-\binom{k}{2}}}{(q;q)_k}, \qquad (17)$$

□

linearization of The Al-Salam-Carlitz I

Proposition

The Al-Salam-Carlitz I polynomials sets given by:

$$U_n^{(a)}(x;q) = (-a)^n q^{\binom{n}{2}} {}_2\phi_1 \left(\begin{matrix} q^{-n}, x^{-1} \\ 0 \end{matrix} ; q; \frac{qx}{a} \right). \qquad (18)$$

The following linearization formula can be written in the following form:

$$(x \ominus 1)_q^i (x \ominus 1)_q^j = \sum_{k=0}^{i+j} h_{i,j,k,a} \times {}_3\phi_1 \left(\frac{q^{i+j-k}}{a} \right) U_k^{(a)}(x;q)$$

The linearization coefficients of the q-Pochhahmmer symbols

Definition

The q-Pochhahmmer polynomial defined by

$$(x;q)_n = (1-x)(1-xq)\cdots(1-xq^{n-1}) = {}_1\phi_0\left(\begin{matrix} q^{-n} \\ - \end{matrix}; q; q^n x\right), \quad (19)$$

Proposition

The basis $(x;q)_n$ fulfills the linearization formula:

$$(x;q)_i(x;q)_j = \sum_{k=0}^{\min(i,j)} (-1)^k q^{\frac{k(k-1)}{2}-ij} \frac{(q;q)_i(q;q)_j}{(q;q)_k(q;q)_{i-k}(q;q)_{j-k}} (x;q)_{i+j-k}. \quad (20)$$

The linearization coefficients of the q-Pochhahmmer symbols

proof:

By applying Theorem 2.2 for the basis $(x;q)_n$ the linearization formula holds:

$$(x;q)_i(x;q)_j = \sum_{k=0}^{i+j} L_{ij}(k)(x;q)_k, \tag{21}$$

where

$$L_{ij}(i+j-k) = (-1)^k q^{\frac{k(k-1)}{2}-ij} \begin{bmatrix} i+j \\ k \end{bmatrix}_q \times \phi^{1:1;1}_{1:0;0}.$$

The linearization coefficients of the q-Pochhahmmer symbols

With the aid of q-Multisum algorithm, $S(k) := L_{ij}(i+j-k)q^{ij}$ satisfies the two order recurrence relation, namely

$$q^k\left(-q^{i+j+k+1}+2q^{i+j}-q^{i+k+1}-q^{j+k+1}+q^{2k+1}+q^{2k+2}-q^k\right)S(k+1)+ S(k)\left(q^i-q^k\right)\left(q^j-q^k\right)\left(q^{i+j}-q^k\right)-\left(q^{k+2}-1\right)q^{2k+1}S(k+2)=0 \tag{22}$$

Then using the command *qrecsolve* of *qsum17* packager (see koepf [1998]) to solve the recurrence relation then we obtain:

$$S(k)=\frac{\left(q^{-i};q\right)_k\left(q^{-j};q\right)_k\left(-q^{i+j}\right)^k q^{-\binom{k}{2}}}{(q;q)_k} \tag{23}$$

□

The linearization coefficient of the Pochhammer polynomials

Remark

In view of the limit relation (2) we can deduce the linearization coefficient of the Pochhammer polynomials $\{(x)_n\}_{n\geq 0}$ *witch was already given in A. Ronveaux and al. [2001]:*

$$(x)_i(x)_j = \sum_{k=0}^{\min(i,j)} (-1)^k \frac{(-i)_k(-j)_k}{k!}(x)_{i+j-k}. \qquad (24)$$

Linearization of big q-Jacobi polynomials

Proposition

The big q-Jacobi polynomials $P_n(x; a, b, c; q)$ define by:

$$P_n(x; a, b, c; q) = {}_3\phi_2\left(\begin{matrix} q^{-n}, abq^{n+1}, x \\ aq, cq \end{matrix}; q; q\right), \tag{25}$$

has the following linearization formula:

$$(x; q)_i (x; q)_j = \sum_{k=0}^{i+j} h_{i,j,k} \times {}_4\phi_3\left(\frac{bq}{c}\right) P_k(x; a, b, c; q) \tag{26}$$

Linearization formula of Stieltjes-Wigert

The linearization formula of Stieltjes-Wigert polynomials,

$$S_i(x;q)S_j(x;q) = \sum_{k=0}^{\min(i,j)} L_{ij}(k) S_k(x,q), \tag{27}$$

where $S_n(x;q) = \frac{1}{(q;q)_n} {}_1\phi_1\left(\begin{matrix} q^{-n} \\ 0 \end{matrix}; q; -q^{n+1}x\right)$, and

$$L_{ij}(i+j-k) = (-1)^k q^{\frac{k(k-1)}{2}} \frac{q^{-2ij}}{(q;q)_k} \begin{bmatrix} i+j \\ j \end{bmatrix}_q \times \phi^{1:1;1}_{1:0;0},$$

Unfortunately we obtain a recurrence relation of order 7.

Thank you for your attention